H2 Mathematics 2016 Paper 2 Revision Solutions

$$f(x) = \frac{ax^2+bx+1}{x+c} = (ax+b-ac)+\frac{1-bc+ac^2}{x+c}$$

1

$ax+b-ac = 2x-1$

$\therefore a = 2$ (ans) and $b-ac=-1 \Rightarrow b = 2c-1$ (shown)

Given $c=1$, $f(x) = 2x-1+\dfrac{2}{x+1}$

(i) Let $y = 2x-1+\dfrac{2}{x+1}$

$(x+1)y = (2x-1)(x+1)+2$

$2x^2+(1-y)x+(1-y)=0$

For all real values of x,

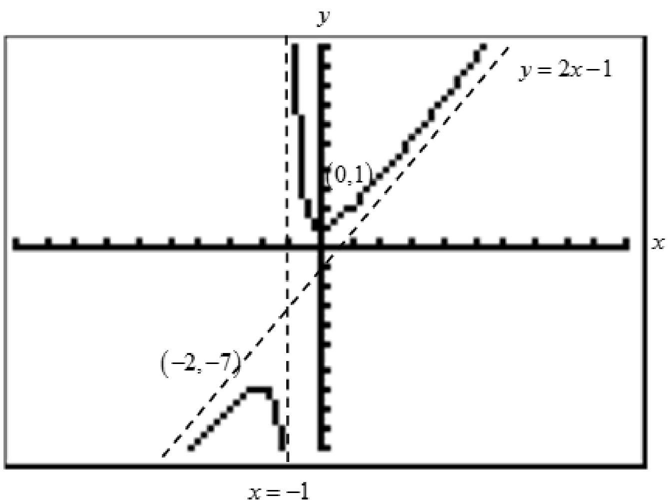
D < 0

$\Rightarrow (1-y)^2 - 4(2)(1-y) < 0$

$\Rightarrow (y+7)(y-1) < 0$

$\Rightarrow \therefore -7 < y < 1$ (ans)

(ii)



f is concaving downwards for $x < -1$.

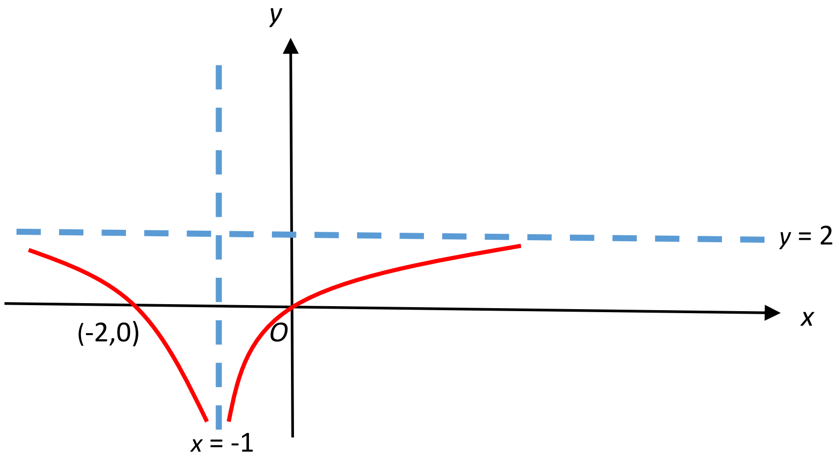
(iii) $2x^2 + x + 1 = (kx + k - 3)(x + 1)$

$$\Rightarrow \frac{2x^2 + x + 1}{x + 1} = kx + k - 3$$

The line $kx + (k - 3)$ passes through the point $(-1, -3)$, which is the intersection of the asymptotes. Since the oblique asymptote passes through the point $(-1, -3)$ and using the graph in (iii), the gradient of the line $kx + k - 3$ has to be more than 2 for the above equation to have 2 real solutions.

Hence, $k > 2$.

(iv)



Graph is concave upwards for x > -1.

2 **(i)** $2x + 2z = 24$, $y + 2z = 9$

Solving the above simultaneous equation,
$z = 12 - x$, $y = 2x - 15$

$$V = xyz = x(2x - 15)(12 - x)$$
$$= -2x^3 + 39x^2 - 180x$$

(ii) $\frac{dV}{dx} = 0$

$-6x^2 + 78x - 180 = 0$

$x^2 - 13x + 30 = 0$

$(x - 10)(x - 3) = 0$

$x = 3 \quad x = 10$

$\frac{d^2V}{dx^2} = -12x + 78$

When $x=3$, $\frac{d^2V}{dx^2}=42>0$,

When $x=10$, $\frac{d^2V}{dx^2}=-42<0$

Hence, when $x=10$, maximum volume = (10)(5)(2)=100

3i Area of $R=\int_{-1}^{0} y\,\mathrm{d}x=\int_{-1}^{0}\left[x^3+1\right]\,\mathrm{d}x=\frac{3}{4}$

$y=1+x^3 \Rightarrow x=(y-1)^{\frac{1}{3}}$

Area of $S=\int_{2}^{b} x\,\mathrm{d}y=\int_{2}^{b}(y\text{-}1)^{\frac{1}{3}}\,\mathrm{d}y=\frac{3}{4}\left[(y-1)^{\frac{4}{3}}\right]_2^b$

$=\frac{3}{4}\left[(b-1)^{\frac{4}{3}}-1\right]$

Equating and solve for b:

$\frac{3}{4}\left[(b-1)^{\frac{4}{3}}-1\right]=\frac{3}{4}$

$\Rightarrow b=1+2^{\frac{3}{4}}=2.68$ (3 s.f.)

3ii For $y=b$, $x=(b-1)^{\frac{1}{3}}=1.1892=k$ (say)

Volume required

$=\pi\left[b^2k-2^2(1)-\int_{1}^{k}(x^3+1)^2\,\mathrm{d}x\right]$

= 3.53π (or 11.1) (unit cube)

4a $\int \frac{\mathrm{e}^t}{(1+3\mathrm{e}^t)^2}\,\mathrm{d}t=\frac{1}{3}\int 3\mathrm{e}^t(1+3\mathrm{e}^t)^{-2}\,\mathrm{d}t$

$=\frac{(1+3\mathrm{e}^t)^{-1}}{-3}+c=-\frac{1}{3(1+3\mathrm{e}^t)}+c$

b

$$\frac{d}{dx}\left(\tan\left(x^2\right)\right) = 2x\sec^2\left(x^2\right)$$

$$\int x^3 \sec^2\left(x^2\right) dx = \frac{1}{2}\int x^2\left[2x\sec^2\left(x^2\right)\right]dx$$

$$= \frac{1}{2}\left[x^2 \tan\left(x^2\right) - \int 2x\tan\left(x^2\right) dx\right]$$

$$= \frac{1}{2}\left[x^2 \tan\left(x^2\right) - \ln\left|\sec\left(x^2\right)\right|\right] + c$$

c

$$\int_0^4 x^2 |x-3|\ dx$$

$$= -\int_0^3 x^2(x-3)\ dx + \int_3^4 x^2(x-3)\ dx$$

$$= -\left[\frac{x^4}{4} - x^3\right]_0^3 + \left[\frac{x^4}{4} - x^3\right]_3^4$$

$$= \frac{27}{2} \quad \text{or} \quad 13.5$$

5i $x = e^{\theta}\cos\theta, \qquad y = \sin\theta + \cos\theta, \qquad 0 \le \theta \le \frac{\pi}{4}$

$$\frac{dx}{d\theta} = e^{\theta}(\cos\theta - \sin\theta), \qquad \frac{dy}{d\theta} = \cos\theta - \sin\theta,$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} / \frac{dx}{d\theta} = e^{-\theta}$$

At $(e^{\theta}\cos\theta, \sin\theta + \cos\theta)$, the equation of the tangent is

$$(y - \sin\theta - \cos\theta) = e^{-\theta}(x - e^{\theta}\cos\theta),$$

Set $\theta = \frac{\pi}{6}$,

at $\left(\frac{\sqrt{3}e^{\frac{\pi}{6}}}{2}, \frac{\sqrt{3}+1}{2}\right)$, the equation of the tangent is $\left(y - \frac{\sqrt{3}}{2} - \frac{1}{2}\right) = e^{-\frac{\pi}{6}}\left(x - \frac{\sqrt{3}e^{\frac{\pi}{6}}}{2}\right)$,

$$y = e^{-\frac{\pi}{6}}x + \frac{1}{2}$$

ii Area under the curve C is

$$A = \int_0^{\frac{\pi}{4}} (\sin\theta + \cos\theta)e^{\theta}(\cos\theta - \sin\theta)\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} e^{\theta}(\cos^2\theta - \sin^2\theta)\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} e^{\theta}\cos 2\theta\, d\theta \qquad \textit{(shown)}$$

$$= 0.68\ \ (2\,\text{d.p.})$$

6i

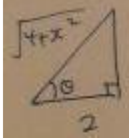
Q6i) let $x = 2\tan\theta$

$$\frac{dx}{d\theta} = 2\sec^2\theta$$

$$\int \frac{4-x^2}{(4+x^2)^2}\,dx = \int \frac{4-(2\tan\theta)^2}{\left(4+(2\tan\theta)^2\right)^2}\, 2\sec^2\theta\, d\theta$$

$$= \int \frac{4-4\tan^2\theta}{16\sec^4\theta}\, 2\sec^2\theta\, d\theta$$

$$= \int \frac{1-\tan^2\theta}{2\sec^2\theta}\, d\theta$$



$$= \frac{1}{2}\int \frac{1}{\sec^2\theta} - \frac{\tan^2\theta}{\sec^2\theta}\, d\theta$$

$$= \frac{1}{2}\int \cos^2\theta - \sin^2\theta\, d\theta$$

$$= \frac{1}{2}\int \cos 2\theta\, d\theta$$

$$= \frac{1}{4}\sin 2\theta + C$$

$$= \frac{2}{4}\sin\theta\cos\theta + C$$

$$= \frac{1}{2}\left(\frac{x}{\sqrt{4+x^2}}\right)\left(\frac{2}{\sqrt{4+x^2}}\right) + C$$

$$= \frac{x}{4+x^2} + C$$

ii) Vol. generated $= \int_{-2}^{1} \pi \frac{4-x^2}{(4+x^2)^2}\,dx - \frac{1}{3}\pi\left(\frac{\sqrt{3}}{5}\right)^2(3)$

$$= \pi\left[\frac{x}{4+x^2}\right]_{-2}^{1} - \frac{3}{25}\pi$$

$$= \pi\left(\frac{1}{5} - \left(\frac{-2}{8}\right)\right) - \frac{3}{25}\pi = \frac{33}{100}\pi$$

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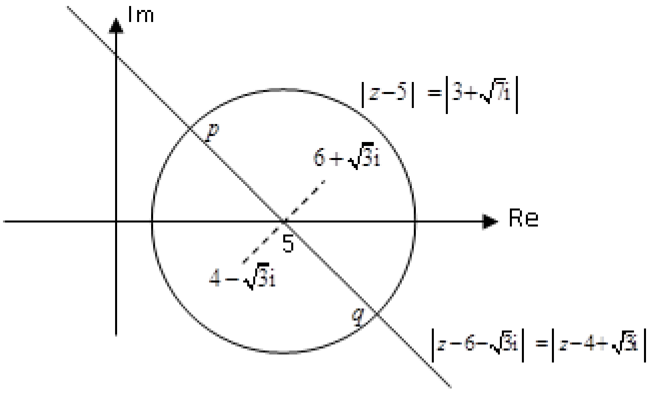

Cartesian equation of (i):

$$(x-5)^2 + y^2 = 4^2 \qquad \text{---- (1)}$$

Gradient of line segment joining $6+\sqrt{3}i$ and $4-\sqrt{3}i$

$= \dfrac{\sqrt{3}-\left(-\sqrt{3}\right)}{6-4} = \sqrt{3}$

Cartesian equation of (ii):

$$y = -\frac{1}{\sqrt{3}}\left(x-5\right) \qquad \text{---- (2)}$$

Using GC to solve (1) & (2), we get

$p = 1.5359 + 2i$

$q = 8.4641 - 2i$

$\therefore \arg(p-q) = \pi - \sin^{-1}\left(\dfrac{4}{8}\right) = \dfrac{5\pi}{6} = 2.62$ (3 s.f.)

Alternative:

$p - q = -6.9282 + 4i$

$\therefore \arg\left(p-q\right) = \pi - \tan^{-1}\dfrac{4}{6.9282} = 2.62$ (3s.f.)

Or using GC,

$\therefore \arg\left(p-q\right) = 2.62$ (3s.f.)

8i

$z^5 - 32 = 0 \Rightarrow z^5 = 32\,e^{i0} = 32e^{i2k\pi}$

$\Rightarrow z = 2e^{2k\pi i/5}$ where $k = 0, \pm 1, \pm 2.$

ii The highest power in the equation $\left(\frac{2w+1}{w}\right)^5 = 32$ is four since the terms with w^5 are canceled out. Hence the equation has only four roots.

$$\left(2+\frac{1}{w}\right)^5 = 32$$

$$\Rightarrow 2+\frac{1}{w} = z = 2e^{2k\pi i/5}$$

$$\Rightarrow \frac{1}{w} = 2e^{2k\pi i/5} - 2 = 2\left(e^{2k\pi i/5} - 1\right)$$

$$\Rightarrow \frac{1}{w_1}+\frac{1}{w_2}+\frac{1}{w_3}+\frac{1}{w_4}$$

$$= 2\left[\left(e^{2\pi i/5}-1\right)+\left(e^{-2\pi i/5}-1\right)+\left(e^{4\pi i/5}-1\right)+\left(e^{-4\pi i/5}-1\right)\right]$$

$$= 2\left[2\cos\frac{2\pi}{5}+2\cos\frac{4\pi}{5}-4\right]$$

$$= 4\left[\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}-2\right] \in \mathbb{R}.$$

Or use GC, $\frac{1}{w_1}+\frac{1}{w_2}+\frac{1}{w_3}+\frac{1}{w_4} = -10.$

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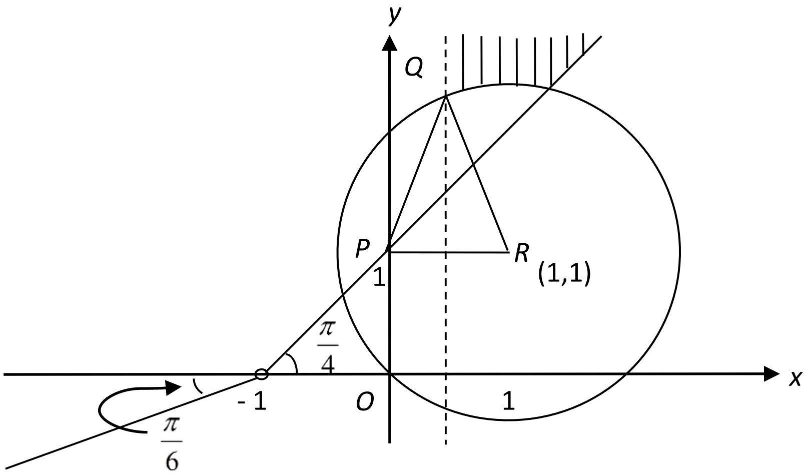

(i) $\quad \left|z-1-\mathrm{i}\right|^2 \geq 2$

$\Rightarrow \left|z-(1+\mathrm{i})\right| \geq \sqrt{2}$

(ii) $\quad \frac{\pi}{12} \leq \arg\left(\frac{z+1}{\sqrt{3}+\mathrm{i}}\right) \leq \pi$

$\Rightarrow \frac{\pi}{12}+\frac{\pi}{6} \leq \arg(z+1) \leq \pi+\frac{\pi}{6}$

$\Rightarrow \frac{\pi}{4} \leq \arg(z+1) \leq \frac{7\pi}{6}$

$PQ=\sqrt{2-\frac{1}{4}}=\frac{\sqrt{7}}{2} \quad ; \quad QR=\sqrt{\frac{7}{4}+\frac{1}{4}}=\sqrt{2}$

$\frac{\pi}{4} \leq \arg\left(z-\mathrm{i}\right) < \frac{\pi}{2}$