Suggested solutions to 2016 Paper 1 (H2 Math)

1) $\dfrac{4x^2+4x-14}{x-4} - (x+3) = \dfrac{4x^2+4x-14-(x-4)(x+3)}{x-4}$

$= \dfrac{(4x^2+4x-14)-(x^2-x-12)}{x-4}$

$= \dfrac{3x^2+5x-2}{x-4}$

$\dfrac{4x^2+4x-14}{x-4} < x+3 \Rightarrow \dfrac{4x^2+4x-14}{x-4} - (x+3) < 0$

$\Rightarrow \dfrac{3x^2+5x-2}{x-4} < 0$

$\Rightarrow \dfrac{(3x-1)(x+2)}{x-4} < 0$

$\therefore \; x < -2$ or $\frac{1}{3} < x < 4$.

2i) $y = 2^{\cos x} \Rightarrow \left.\dfrac{dy}{dx}\right|_{x=0} = 0$

$\left.\dfrac{dy}{dx}\right|_{x=\frac{\pi}{2}} \simeq -0.693$

ii) when $x=0$, tangent is a horizontal line $\Rightarrow$ $y=2$.

$y = 2^{\cos 0}$
$= 2$

when $x=\frac{\pi}{2}$, $y = 2^{\cos\frac{\pi}{2}} = 1$

$\therefore$ eqn of tangent: $y-1 = -0.693\left(x-\frac{\pi}{2}\right)$

$y = -0.693x + 2.01$

3)

$y=x^4$	$y=f(x)$
$(0,0)$ $\rightarrow$	(a,b)
	$(0,c)$.

$f(x) = k(x-l)^4 + m$.

when $x=0$, $f(0)=c$

$\Rightarrow c = kl^4 + m$. (1)

Also turning point of $f(x)$

$= (l, m) = (a,b)$.

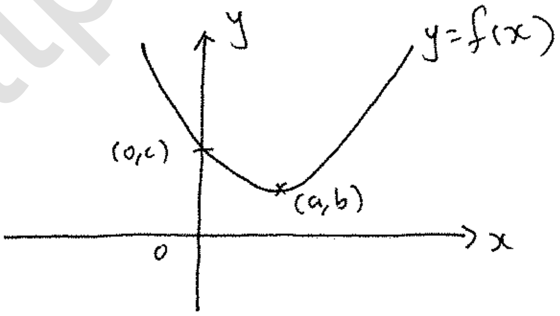
Comparing coefficient,

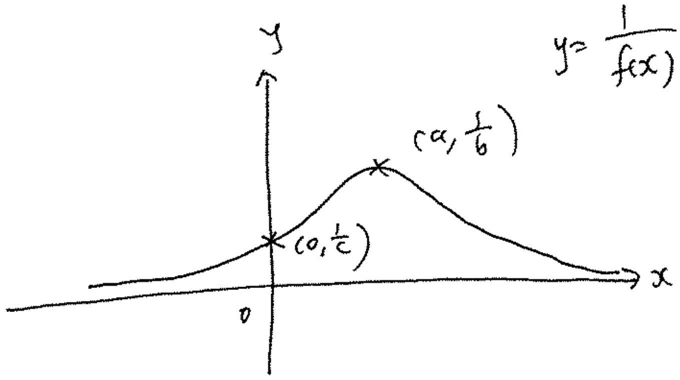
$l=a$, $m=b$ (2).

sub (2) into (1).

$c = ka^4 + b \Rightarrow k = \frac{c-b}{a^4}$

$\therefore f(x) = \frac{c-b}{a^4}(x-a)^4 + b$.





Q4i) $\frac{a+8d}{a+3d} = r^3$ ①, $\frac{a+11d}{a+3d} = r^{10}$. ②

$\Rightarrow 1+\frac{5d}{a+3d} = r^3$

$\Rightarrow \frac{a+3d}{5d} = \frac{1}{r^3-1}$

$\Rightarrow a = \frac{5d}{r^3-1} - 3d$ ③.

Sub ③ into ②.

$$\frac{\frac{5d}{r^3-1} - 3d + 11d}{\frac{5d}{r^3-1} - 3d + 3d} = r^{10} \Rightarrow \frac{\frac{5d}{r^3-1} + 8d}{\frac{5d}{r^3-1}} = r^{10}.$$

$\Rightarrow \frac{5+8(r^3-1)}{5} = r^{10}.$

$\Rightarrow 8r^3 - 3 = 5r^{10} \Rightarrow 5r^{10} - 8r^3 + 3 = 0.$

Using G.C, $r = 0.74$ (2 d.p) or $r = 1$ (rejected)

ii) $u_n = b(0.74)^{n-1}$.

Sum required $= S_\infty - S_n = \frac{b}{1-0.74} - \frac{b(1-0.74^n)}{1-0.74}$

$= 3.85\,b\,(0.74^n)$

5) $\underset{\sim}{u} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}, \quad \underset{\sim}{v} = \begin{pmatrix} a \\ 0 \\ b \end{pmatrix}$

i) $\underset{\sim}{u} + \underset{\sim}{v} = \begin{pmatrix} 2+a \\ -1 \\ 2+b \end{pmatrix}, \quad \underset{\sim}{u} - \underset{\sim}{v} = \begin{pmatrix} 2-a \\ -1 \\ 2-b \end{pmatrix}$

$$(\underset{\sim}{u}+\underset{\sim}{v}) \times (\underset{\sim}{u}-\underset{\sim}{v}) = \begin{pmatrix} 2+a \\ -1 \\ 2+b \end{pmatrix} \times \begin{pmatrix} 2-a \\ -1 \\ 2-b \end{pmatrix}$$

$$= \begin{pmatrix} (b-2) - (-2-b) \\ -(2+a)(2-b) + (2+b)(2-a) \\ (-2-a) - (-2+a) \end{pmatrix}$$

$$= \begin{pmatrix} 2b \\ 4b-4a \\ -2a \end{pmatrix}$$

ii) $2b = -2a \Rightarrow b = -a$.

$$\therefore (\underset{\sim}{u}+\underset{\sim}{v}) \times (\underset{\sim}{u}-\underset{\sim}{v}) = \begin{pmatrix} -2a \\ -8a \\ -2a \end{pmatrix}$$

since $(\underset{\sim}{u}+\underset{\sim}{v}) \times (\underset{\sim}{u}-\underset{\sim}{v})$ is a unit vector,

$$\left| \begin{pmatrix} -2a \\ -8a \\ -2a \end{pmatrix} \right| = 1 \Rightarrow \sqrt{(-2a)^2 + (-8a)^2 + (-2a)^2} = 1$$

$$\Rightarrow 72a^2 = 1$$

$$\Rightarrow a = \pm \frac{1}{\sqrt{72}} = \pm \frac{1}{6\sqrt{2}}.$$

iii) $(\underset{\sim}{u}+\underset{\sim}{v})\cdot(\underset{\sim}{u}-\underset{\sim}{v})=0$

$$\begin{pmatrix}2+a\\-1\\2+b\end{pmatrix}\cdot\begin{pmatrix}2-a\\-1\\2-b\end{pmatrix}=0$$

$$(2^2-a^2)+(-1)^2+(2^2-b^2)=0.$$

$$9-a^2-b^2=0.$$

$$a^2+b^2=9$$

since $|\underset{\sim}{v}|=\left|\begin{pmatrix}a\\0\\b\end{pmatrix}\right|=\sqrt{a^2+b^2}$

$$|\underset{\sim}{v}|=\sqrt{9}=3.$$

Q6i) Let $P(n)$ be the statement

$$\sum_{r=1}^{n} r(r^2+1)=\frac{1}{4}n(n+1)(n^2+n+2),\quad n\in\mathbb{Z}^+.$$

when $n=1$

LHS : $1(1^2+1)=2$

RHS : $\frac{1}{4}(1)(2)(4)=2\quad=\text{LHS}$

$\therefore P(1)$ is true.

Assume $P(k)$ is true for some $k\in\mathbb{Z}^+$, i.e.

$$\sum_{r=1}^{k} r(r^2+1)=\frac{1}{4}k(k+1)(k^2+k+2)$$

We want to prove

$$\sum_{r=1}^{k+1} r(r^2+1)=\frac{1}{4}(k+1)(k+2)\left[(k+1)^2+k+1+2\right]$$

$$=\frac{1}{4}(k+1)(k+2)(k^2+3k+4)$$

LHS: $\sum_{r=1}^{k+1} r(r^2+1) = \sum_{r=1}^{k} r(r^2+1) + (k+1)\left[(k+1)^2+1\right]$

$= \frac{1}{4}k(k+1)\left[k^2+k+2\right] + (k+1)\left[(k+1)^2+1\right]$

$= \frac{1}{4}(k+1)\left[k^3+k^2+2k+4(k+1)^2+4\right]$

$= \frac{1}{4}(k+1)\left[k^3+5k^2+10k+8\right]$

$= \frac{1}{4}(k+1)(k+2)(k^2+3k+4)$ = RHS.

$\therefore P(k)$ is true $\Rightarrow P(k+1)$ is true.

Since $P(1)$ is true, $P(k)$ is true $\Rightarrow P(k+1)$ is true,
by Mathematical Induction, $P(n)$ is true for all $n \in \mathbb{Z}^+$.

ii) $u_1 = u_0 + 1^3 + 1 = 4$

$u_2 = u_1 + 2^3 + 2 = 14$

$u_3 = u_2 + 3^3 + 3 = 44$.

iii) $\sum_{r=1}^{n}(u_r - u_{r-1}) = \sum_{r=1}^{n}(r^3+r) = \sum_{r=1}^{n} r(r^2+1)$

LHS: $\sum_{r=1}^{n}(u_r - u_{r-1})$

$= \cancel{u_1} - u_0$
$+ \cancel{u_2} - \cancel{u_1}$
$+ \cancel{u_3} - \cancel{u_2}$
$+ \ldots$
$+ \cancel{u_{n-1}} - \cancel{u_{n-2}}$
$+ u_n - \cancel{u_{n-1}}$

$= u_n - u_0$

$= u_n - 2$.

$\therefore u_n - 2 = \frac{1}{4}n(n+1)(n^2+n+2)$

$u_n = \frac{1}{4}n(n+1)(n^2+n+2) + 2$ //.

Q7a) let $w = -1+5i$

$(-1+5i)^2 + (-1-8i)(-1+5i) + (-17+7i)$

$= (1-10i-25) + (1+3i+40) + (-17+7i)$

$= 0$

$\therefore$ $w = -1+5i$ is a root.

let $a+bi$ be 2nd root.

$(w-(-1+5i))(w-(a+bi)) = w^2 + (-1-8i)w + (-17+7i)$

$w^2 + w[-(a+bi)-(-1+5i)] + (-1+5i)(a+bi) = w^2 + (-1-8i)w + (-17+7i)$

$w^2 + w(1-a + (-5-b)i) + (-1+5i)(a+bi) = w^2 + (-1-8i)w + (-17+7i)$

Compare coeff of w term.

$$\left.\begin{aligned} 1-a &= -1 \\ -5-b &= -8 \end{aligned}\right\} \quad \begin{aligned} a &= 2 \\ b &= 3. \end{aligned}$$

$\therefore$ 2nd root is $2+3i$.

b) Since $z = 1+ai$ is a root,

$(1+ai)^3 - 5(1+ai)^2 + 16(1+ai) + k = 0.$

$(1+3ai-3a^2-a^3i) - 5(1+2ai-a^2) + (16+16ai) + k = 0.$

Compare imaginary part

$3a - a^3 - 10a + 16a = 0 \Rightarrow 9a - a^3 = 0 \Rightarrow a = 3.$

Compare real part

$1 - 3a^2 - 5 + 5a^2 + 16 + k = 0 \Rightarrow k = -30.$

Q8i) $f(x) = \tan(ax+b)$

$y = \tan(ax+b)$

$f'(x) = a\sec^2(ax+b)$

$= a[\tan^2(ax+b)+1]$

$= a + ay^2$

$f''(x) = a\left(2y\frac{dy}{dx}\right) = 2ay(a+ay^2) = 2a^2y + 2a^2y^3$

$f'''(x) = a\left[2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2}\right]$

$= 2a(a+ay^2)^2 + 2ay(2a^2y + 2a^2y^3)$

$= 2a(a^2+2a^2y^2+a^2y^4) + 2ay(2a^2y+2a^2y^3)$

$= 2a^3 + 4a^3y^2 + 2a^3y^4 + 4a^3y^2 + 4a^3y^4$

$= 2a^3 + 8a^3y^2 + 6a^3y^4.$

ii) when $x=0$, $b=\frac{\pi}{4}$, $y = \tan(\frac{\pi}{4}) = 1$

$f'(0) = 2a$

$f''(0) = 4a^2$

$f'''(0) = 16a^3$

$\therefore f(x) = 1 + 2ax + \frac{4a^2}{2!}x^2 + \frac{16a^3}{3!}x^3 + \ldots$

$= 1 + 2ax + 2a^2x^2 + \frac{8}{3}a^3x^3 + \ldots$

iii) $y = \tan 2x$

$$\frac{dy}{dx} = 2\sec^2 2x$$

$$\frac{d^2y}{dx^2} = 8\sec 2x(\sec 2x \tan 2x)$$

$$= 8\sec^2 2x \tan 2x$$

$$= 8(\tan^2 2x + 1)\tan 2x$$

$$= 8\tan^3 2x + 8\tan 2x.$$

$$\frac{d^3y}{dx^3} = 48\tan^2 2x(\sec^2 2x) + 16\sec^2 2x$$

when $x=0$, $y=0$, $\frac{dy}{dx} = 2$, $\frac{d^2y}{dx^2} = 0$,

$$\frac{d^3y}{dx^3} = 16.$$

$$\therefore \tan 2x = 2x + \frac{8}{3}x^3 + \ldots$$

Q9) a) $y = \frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \frac{d^2x}{dt^2}.$

$\therefore$ New D.E $\quad \frac{dy}{dt} + 2y = 10.$

$$\frac{dy}{dt} = 10 - 2y$$

b) $\int \frac{1}{10-2y}\, dy = \int 1\, dt$

$$-\frac{1}{2}\ln|10-2y| = t + c$$

$$\ln|10-2y| = -2t - 2c$$

$$|10-2y| = e^{-2t-2c}$$

$$10-2y = \pm e^{-2t-2c} = Ae^{-2t}, \text{ where } A = \pm e^{-2c}$$

$$y = \frac{10-Ae^{-2t}}{2}.$$

when $t=0$, $y=0$.

$$0 = \frac{10-A}{2} \Rightarrow A = 10.$$

$$\therefore y = 5-5e^{-2t}. \Rightarrow \frac{dx}{dt} = 5-5e^{-2t} \Rightarrow x = 5t + \frac{5}{2}e^{-2t} + f$$

when $t=0$, $x=0 \Rightarrow f = -\frac{5}{2}$.

$$x = 5t + \frac{5}{2}e^{-2t} - 5/2.$$

ii) $$\frac{d^2x}{dt^2} = 10 - 5\sin\left(\tfrac{1}{2}t\right)$$

$$\frac{dx}{dt} = 10t + 10\cos\left(\tfrac{1}{2}t\right) + c$$

$$x = \frac{10t^2}{2} + 20\sin\left(\tfrac{1}{2}t\right) + ct + d$$

when $t=0$, $x=0$.

$d=0$.

when $t=0$ $\frac{dx}{dt}=0$ $\Rightarrow c = -10$.

$$\therefore x = 5t^2 + 20\sin\left(\tfrac{1}{2}t\right) - 10t.$$

iii) when $x=5$,

from 1st model

$5 = 5t + \frac{5}{2}e^{-2t} - \frac{5}{2}$

$\Rightarrow 5t + \frac{5}{2}e^{-2t} - \frac{15}{2} = 0$

using G.C (graphical).

$t \approx 1.47$ s.

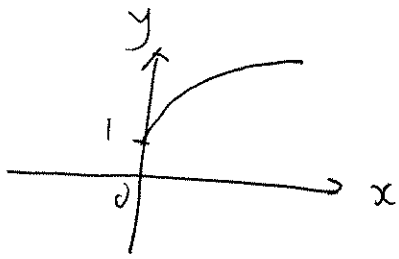
from 2nd model

$5 = 5t^2 + 20\sin(\frac{1}{2}t) - 10t$

using G.C (graphical)

$t \approx 1.05$ s.

Q10a i) $f(x) = 1 + \sqrt{x}$, $x \geq 0$.



$y = 1 + \sqrt{x}$

$x = (y-1)^2$

$\therefore f^{-1}(x) = (x-1)^2$, $x \in [1, \infty)$

ii) If $ff(x) = x \Rightarrow \sqrt{1+\sqrt{x}} + 1 = x$

$\sqrt{1+\sqrt{x}} = x - 1 \Rightarrow 1 + \sqrt{x} = (x-1)^2$

$\Rightarrow x = [(x-1)^2 - 1]^2$

$\Rightarrow x = (x-1)^4 - 2(x-1)^2 + 1$

$x = x^4 - 4x^3 + 6x^2 - 4x + 1 - 2x^2 + 4x - 2 + 1$

$\Rightarrow x^4 - 4x^3 + 4x^2 - x = 0$

$x(x^3 - 4x^2 + 4x - 1) = 0$

$x = 0$ or $x^3 - 4x^2 + 4x - 1 = 0$

(rej. since $ff(0) \neq 0$)

Using G.C

$x^3 - 4x^2 + 4x - 1 = 0 \Longrightarrow x = 2.62$

$ff(x) = x \Rightarrow f^{-1}ff(x) = f^{-1}(x)$

$\Rightarrow f(x) = f^{-1}(x)$

$\therefore x = 2.62$ satisfies $f(x) = f^{-1}(x)$

b) i) $g(4) = 2 + g(2) = 2 + (2 + g(1))$

$= 4 + (1 + g(0))$

$= 6$.

$g(7) = 1 + g(6) = 1 + (2 + g(3)) = 3 + g(3)$

$= 3 + (1 + g(2)) = 4 + g(2) = 4 + 2 + g(1)$

$= 6 + g(1) = 6 + 1 + g(0) = 8$.

$g(12) = 2 + g(6) = 2 + 7 = 9$.

ii) No. $g(7) = g(8) = 8$.

Q11) a) normal vector of p $= \begin{pmatrix}1\\2\\0\end{pmatrix} \times \begin{pmatrix}0\\4\\-2\end{pmatrix} = \begin{pmatrix}-4\\2\\4\end{pmatrix}$

Since direction vector of l, $\begin{pmatrix}-2\\1\\2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-4\\2\\4\end{pmatrix}$

i.e. parallel to normal of p,

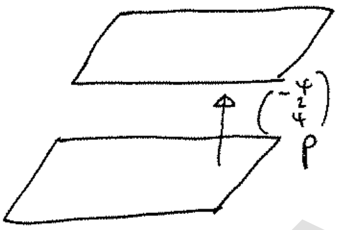
$l \perp p$.

when $a=0$.

$$\begin{pmatrix}-1-2t\\t\\1+2t\end{pmatrix} = \begin{pmatrix}1+\lambda\\-3+2\lambda+4\mu\\2-2\mu\end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} -2t-\lambda &= 2\\ t-2\lambda-4\mu &= -3\\ 2t+2\mu &= 1\end{aligned}\right\} \text{using G.C}$$

$\lambda = -\frac{8}{9}$, $\mu = \frac{19}{18}$, $t = -5/9$

b) 

eqn of 2 planes are

$\pi_1 : \underset{\sim}{r} \cdot \begin{pmatrix}-4\\2\\4\end{pmatrix} = k_1$

$\pi_2 : \underset{\sim}{r} \cdot \begin{pmatrix}-4\\2\\4\end{pmatrix} = k_2$.

Scalar product eqn of p: $\underset{\sim}{r} \cdot \begin{pmatrix}-4\\2\\4\end{pmatrix} = \begin{pmatrix}-4\\2\\4\end{pmatrix} \cdot \begin{pmatrix}1\\-3\\2\end{pmatrix} = -2$.

dist. between p & O $= \dfrac{|-2|}{\left|\begin{pmatrix}-4\\2\\4\end{pmatrix}\right|} = \dfrac{1}{3}$.

dist. between π_1 & O $= \frac{1}{3} + 12 = \frac{37}{3}$

dist. between π_2 & O $= 12 - \frac{1}{3} = \frac{35}{3}$.

$\Rightarrow \dfrac{|k_1|}{6} = \dfrac{37}{3} \Rightarrow |k_1| = 74$ $\qquad \dfrac{|k_2|}{6} = \dfrac{35}{3} \Rightarrow |k_2| = 70$.

Since Π_1 & p are on same side with respect to O,

$k_1 = -74$

Π_2 & p are on opposite side with respect to O,

$k_2 = 70$.

$\therefore \Pi_1: \ -4x + 2y + 4z = -74$

$\Rightarrow \ -2x + y + 2z = -37$

$\Pi_2: \ -4x + 2y + 4z = 70$

$\Rightarrow \ -2x + y + 2z = 35$

(ii) For l to lie on $p \Rightarrow l \parallel p$ &

$\begin{pmatrix} a-1 \\ a \\ a+1 \end{pmatrix}$ lies on p.

normal of $p = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} a \\ 4 \\ -2 \end{pmatrix}$

$= \begin{pmatrix} -4 \\ 2 \\ 4-2a \end{pmatrix}$.

$\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 2 \\ 4-2a \end{pmatrix} = 0 \Rightarrow 8 + 2 + 8 - 4a = 0$

$a = \frac{9}{2}$.