2016 HCI Prelim Paper 1 Solutions

Qn	Solution
1	* Let P_n be statement $U_n = \sin(nx)$ for all $n \in \mathbb{Z}^+$. When $n=1$, LHS $= U_1 = \sin x$, RHS $= \sin x$ $\therefore P_1$ is true. * Assume P_k is true for some $k \in \mathbb{Z}^+$, i.e. $U_k = \sin(kx)$. Want to prove that P_{k+1} is true, i.e. $U_{k+1} = \sin(k+1)x$. LHS $= U_{k+1}$ $= U_k + 2\cos\dfrac{(2k+1)x}{2}\sin\dfrac{x}{2}$ $= \sin(kx) + 2\cos\left(\dfrac{2k+1}{2}\right)x\sin\left(\dfrac{1}{2}\right)x$ $= \sin(kx) + \sin(k+1)x - \sin(kx)$ $= \sin(k+1)x =$ RHS *Since P_1 is true, P_k is true implies P_{k+1} is true, by MI P_n is true for all $n \in \mathbb{Z}^+$.
2	$\dfrac{2}{4(x+1)^2+1} > 1$ $\dfrac{-(2x+1)(2x+3)}{4(x+1)^2+1} > 0$ Since $4(x+1)^2+1>0$ for all x, $(2x+1)(2x+3)<0$ $\therefore -\dfrac{3}{2} < x < -\dfrac{1}{2}$ $\displaystyle\int_{-1}^{\frac{\sqrt{3}}{2}-1}\left\|1-\frac{2}{4(x+1)^2+1}\right\|\mathrm{d}x$ $\displaystyle= \int_{-1}^{-\frac{1}{2}}\left(-1+\frac{2}{4(x+1)^2+1}\right)\mathrm{d}x + \int_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}-1}\left(1-\frac{2}{4(x+1)^2+1}\right)\mathrm{d}x$ $= \left[-x+\tan^{-1}(2x+2)\right]_{-1}^{-\frac{1}{2}} + \left[x-\tan^{-1}(2x+2)\right]_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}-1}$ $= \left[\dfrac{1}{2}+\tan^{-1}1-1\right] + \left[\dfrac{\sqrt{3}}{2}-1-\tan^{-1}\sqrt{3}+\dfrac{1}{2}+\tan^{-1}1\right]$ $= \dfrac{\pi}{6}+\dfrac{\sqrt{3}}{2}-1$

3 (a)	$\overrightarrow{OP} = \underset{\sim}{a} + 3\overrightarrow{AB} = \underset{\sim}{a} + 3(\underset{\sim}{b} - \underset{\sim}{a}) = 3\underset{\sim}{b} - 2\underset{\sim}{a}$ $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = 2\underset{\sim}{a} - (3\underset{\sim}{b} - 2\underset{\sim}{a}) = 4\underset{\sim}{a} - 3\underset{\sim}{b}$ $l_{PQ} : \underset{\sim}{r} = 2\underset{\sim}{a} + \lambda(4\underset{\sim}{a} - 3\underset{\sim}{b}),\ \lambda \in \mathbb{R}$ $l_{OB} : \underset{\sim}{r} = \mu\underset{\sim}{b},\ \mu \in \mathbb{R}$ At point of intersection, $2\underset{\sim}{a} + \lambda(4\underset{\sim}{a} - 3\underset{\sim}{b}) = \mu\underset{\sim}{b}$ Comparing coefficients of $\underset{\sim}{a}$ and $\underset{\sim}{b}$, $\lambda = -\frac{1}{2},\ \mu = \frac{3}{2}$ $\therefore$ position vector of the point of intersection $= \frac{3}{2}\underset{\sim}{b}$
(b)	$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{pmatrix} 2 \\ -4 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \Rightarrow \underset{\sim}{n} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ Let F be the foot of perpendicular. <u>Method 1</u> $l_{FC} : \underset{\sim}{r} = \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} + s\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix},\ s \in \mathbb{R}\ ,\ \Pi_{OAB} : \underset{\sim}{r} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 0$ $\left[\begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} + s\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right] \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 0$ $-6 + 4 + s(1 + 4 + 1) = 0$ $s = \frac{1}{3}$ $\therefore \overrightarrow{OF} = \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 0+1 \\ 9-2 \\ 12+1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ 7 \\ 13 \end{pmatrix}$ $\therefore F\left(\frac{1}{3}, \frac{7}{3}, \frac{13}{3}\right)$

	<u>Method 2</u> $\overrightarrow{FC} = \left(\overrightarrow{OC} \cdot \hat{\underset{\sim}{n}}\right)\hat{\underset{\sim}{n}}$ $= \left(\begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} \cdot \frac{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}{\left\| \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\|} \right) \frac{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}{\left\| \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\|}$ $= \frac{-6+4}{\sqrt{6}} \frac{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}{\sqrt{6}}$ $= -\frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ $\overrightarrow{OF} = \overrightarrow{OC} + \overrightarrow{CF}$ $= \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 0+1 \\ 9-2 \\ 12+1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ 7 \\ 13 \end{pmatrix}$ $\therefore F\left(\frac{1}{3}, \frac{7}{3}, \frac{13}{3}\right)$
4	<u>Method 1</u> $\frac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}} = \frac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}} \times \frac{\sqrt{n^2+2n}+\sqrt{n^2-1}}{\sqrt{n^2+2n}+\sqrt{n^2-1}}$ $= \frac{(2n+1)\left(\sqrt{n^2+2n}-\sqrt{n^2-1}\right)}{\left(n^2+2n\right)-\left(n^2-1\right)}$ $= \sqrt{n^2+2n}-\sqrt{n^2-1}$

	Method 2 $\left(\sqrt{n^2+2n}-\sqrt{n^2-1}\right)\left(\sqrt{n^2+2n}+\sqrt{n^2-1}\right)$ $=\left(n^2+2n-\left(n^2-1\right)\right)$ $=2n+1$ $\therefore \dfrac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}}=\sqrt{n^2+2n}-\sqrt{n^2-1}$
	$\displaystyle\sum_{n=1}^{N}\frac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}}$ $\displaystyle=\sum_{n=1}^{N}\left(\sqrt{n^2+2n}-\sqrt{n^2-1}\right)$ $=\left[\begin{array}{c}\sqrt{3}-\sqrt{0}\\ +\sqrt{8}-\sqrt{3}\\ \dots\\ +\sqrt{N^2+2N}-\sqrt{N^2-1}\end{array}\right]$ $=\sqrt{N^2+2N}$
(a)	Replace n by $n+1$, $\displaystyle\sum_{n=2}^{N}\frac{2n-1}{\sqrt{n^2-1}+\sqrt{n(n-2)}}$ $\displaystyle=\sum_{n=1}^{N-1}\frac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}}$ $=\sqrt{(N-1)^2+2(N-1)}$ $=\sqrt{N^2-1}$
(b)	Notice that $\sqrt{n^2+2n}>n$ and $\left(\sqrt{n^2-1}\right)^2-(n-1)^2=2n-2\geq 0$. $\Rightarrow \sqrt{n^2-1}\geq n-1$ $\Rightarrow \sqrt{n^2+2n}+\sqrt{n^2-1}>2n-1$ $\Rightarrow \dfrac{1}{\sqrt{n^2+2n}+\sqrt{n^2-1}}<\dfrac{1}{2n-1}$ $\displaystyle\therefore \sum_{n=1}^{N}\frac{2n+1}{2n-1}>\sum_{n=1}^{N}\frac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}}=\sqrt{N^2+2N}$

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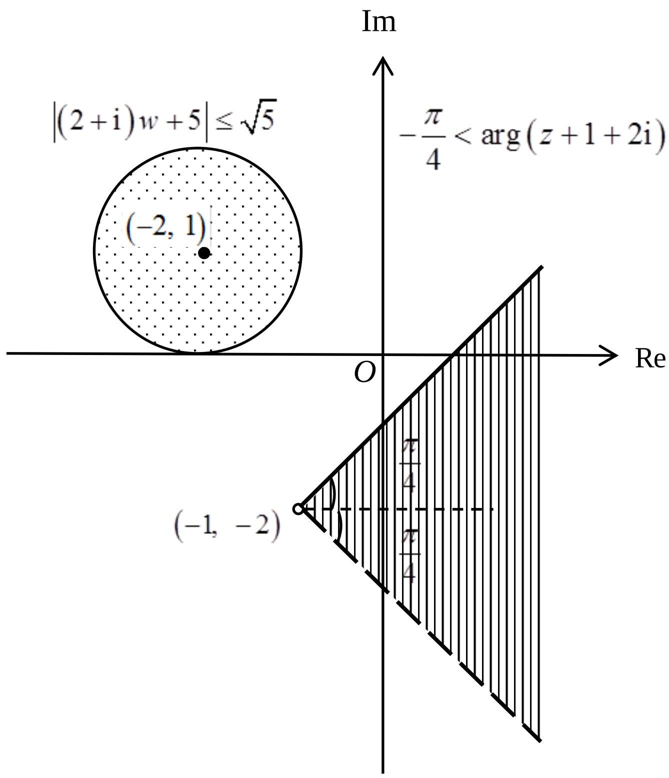
$\left|(2+\mathrm{i})w+5\right| \le \sqrt{5}$

$\left|2+\mathrm{i}\right|\left|w+\dfrac{5}{2+\mathrm{i}}\right| \le \sqrt{5}$

$\left|w+2-\mathrm{i}\right| \le 1 \Rightarrow$ circle centre $(-2,\ 1)$, radius 1



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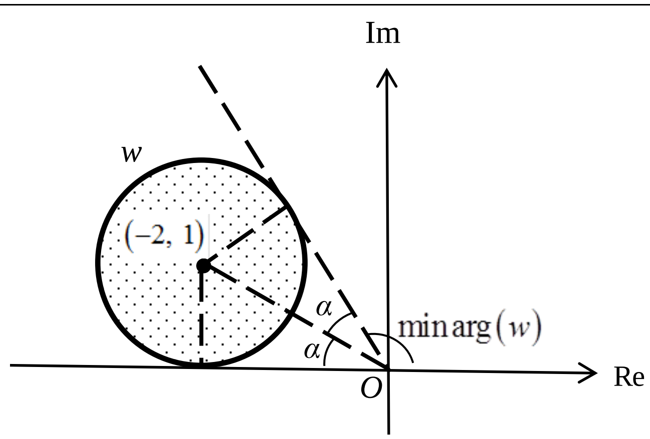
<tr><td>(i)</td><td>



$\alpha = \sin^{-1}\dfrac{1}{\sqrt{(-2)^2+1^2}} = \sin^{-1}\dfrac{1}{\sqrt{5}}$

$\min\arg(w) = \pi - 2\alpha = 2.2143 = 2.21(3\text{sf})$

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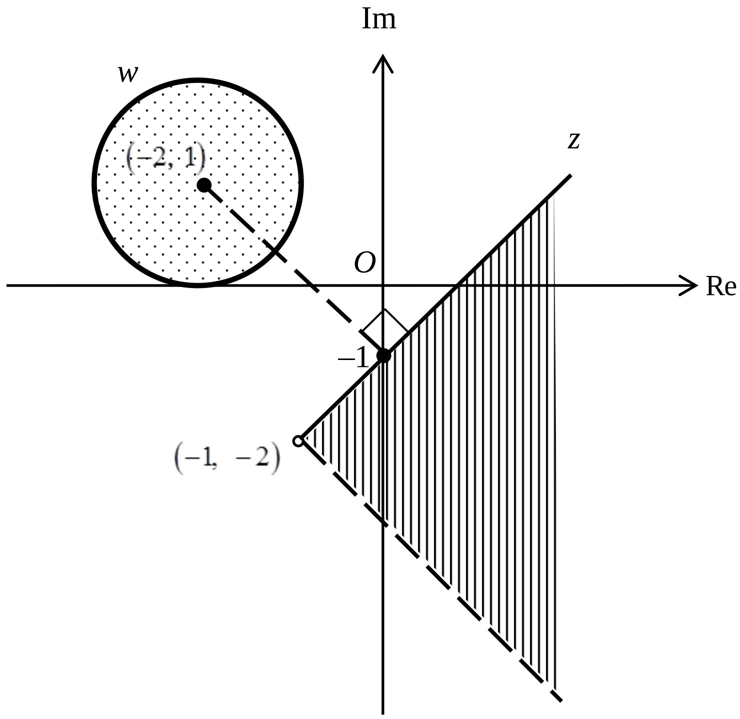
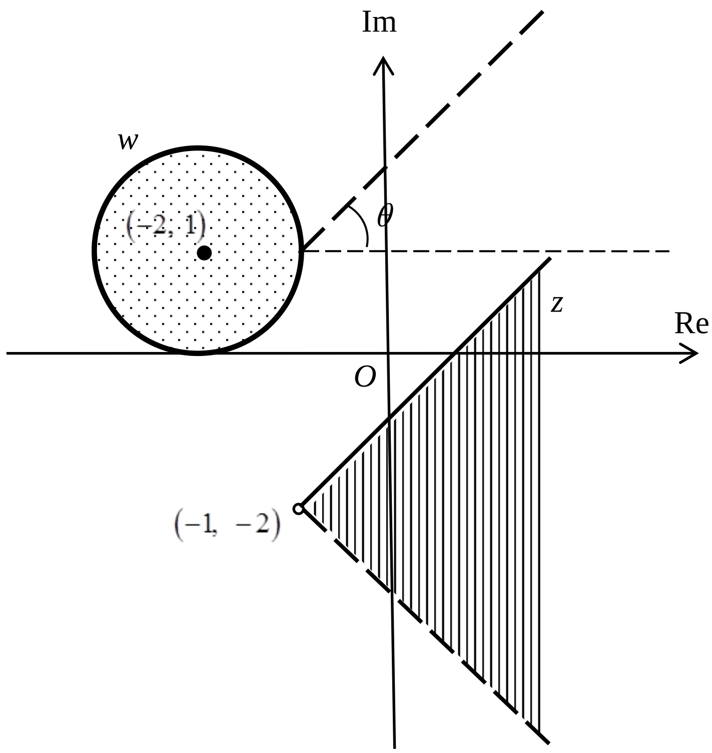
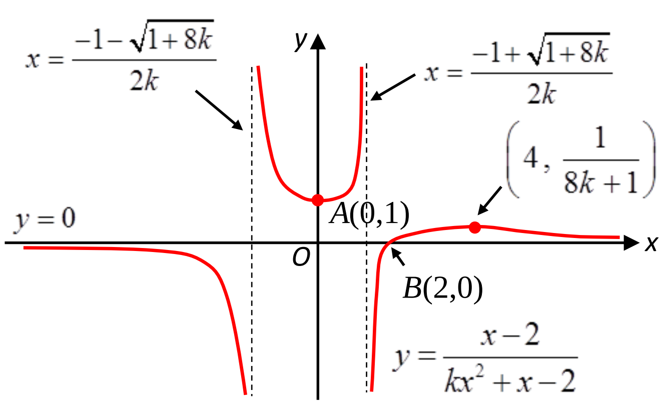
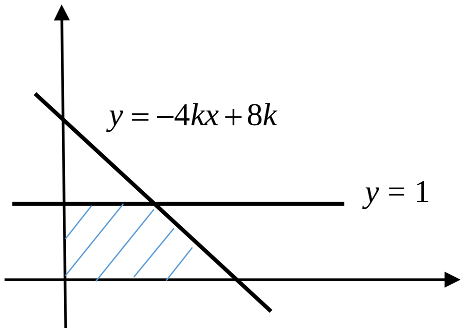
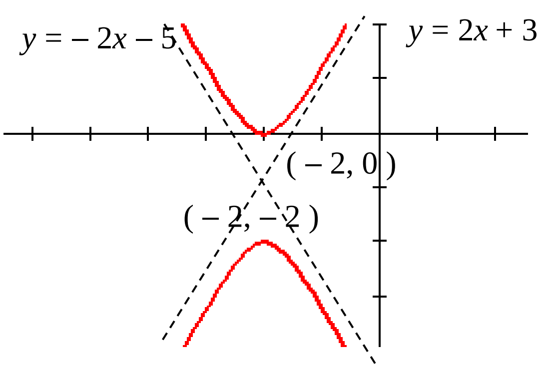
(ii)	 $\min\lvert z-w\rvert = 2\sqrt{2}-1$
(iii)	 $\theta = \frac{\pi}{4}$

6 (i)	$\overrightarrow{OD}=\begin{pmatrix}0\\0\\2\end{pmatrix},\ \overrightarrow{OE}=\begin{pmatrix}3\\1.5\\2\end{pmatrix}$ $\overrightarrow{DE}=\begin{pmatrix}3\\1.5\\2\end{pmatrix}-\begin{pmatrix}0\\0\\2\end{pmatrix}=\begin{pmatrix}3\\1.5\\0\end{pmatrix}=1.5\begin{pmatrix}2\\1\\0\end{pmatrix}$ $l_{DE}: \underset{\sim}{r}=\begin{pmatrix}0\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\0\end{pmatrix},\ \lambda\in\mathbb{R}$
(ii)	$\overrightarrow{AD}=\begin{pmatrix}0\\0\\2\end{pmatrix}-\begin{pmatrix}3\\0\\0\end{pmatrix}=\begin{pmatrix}-3\\0\\2\end{pmatrix}$ $\overrightarrow{DE}\times\overrightarrow{AD}=1.5\begin{pmatrix}2\\1\\0\end{pmatrix}\times\begin{pmatrix}-3\\0\\2\end{pmatrix}=1.5\begin{pmatrix}2\\-4\\3\end{pmatrix}\Rightarrow \underset{\sim}{n}=\begin{pmatrix}2\\-4\\3\end{pmatrix}$ $\Pi_{ADE}: \underset{\sim}{r}\cdot\begin{pmatrix}2\\-4\\3\end{pmatrix}=\begin{pmatrix}0\\0\\2\end{pmatrix}\cdot\begin{pmatrix}2\\-4\\3\end{pmatrix}=6$ $\therefore 2x-4y+3z=6$
(iii)	$\underset{\sim}{n}_{OABC}=\begin{pmatrix}0\\0\\1\end{pmatrix},\ \underset{\sim}{n}_{ADE}=\begin{pmatrix}2\\-4\\3\end{pmatrix}$ angle between planes $=\cos^{-1}\dfrac{\left\|\begin{pmatrix}0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}2\\-4\\3\end{pmatrix}\right\|}{\left\|\begin{pmatrix}0\\0\\1\end{pmatrix}\right\|\left\|\begin{pmatrix}2\\-4\\3\end{pmatrix}\right\|}$ $=\cos^{-1}\dfrac{3}{\sqrt{4+16+9}}$ $=\cos^{-1}\dfrac{3}{\sqrt{29}}$ $=56.1^\circ(1\text{dp})$

	Angle $= 180° - 2\cos^{-1}\dfrac{3}{\sqrt{29}}$ $= 67.7°$ (1 d.p.)
7 (i)	$\dfrac{dy}{dx} = \dfrac{(kx^2+x-2)-(x-2)(2kx+1)}{(kx^2+x-2)^2} = \dfrac{-kx^2+4kx}{(kx^2+x-2)^2}$ When $x=0$, $\dfrac{dy}{dx} = \dfrac{0}{(-2)^2} = 0$ and $y = \dfrac{-2}{-2} = 1$ Hence required equation of tangent is $y = 1$.
(ii)	For axial intercepts, when $y=0$, $x=2$. when $x=0$, $y=1$. For vertical asymptotes, $kx^2+x-2=0$ $\therefore\ x = \dfrac{-1\pm\sqrt{1+8k}}{2k}$ For turning points, $\dfrac{dy}{dx}=0$ $-kx^2+4kx=0$ $-kx(x-4)=0$ $\therefore\ x=0$ or $x=4$ 
(iii)	At $x=2$, $\dfrac{dy}{dx} = \dfrac{-4k+8k}{(4k)^2} = \dfrac{4k}{16k^2} = \dfrac{1}{4k}$ $\therefore$ gradient of normal $= -4k$ Hence required equation of normal is $y-0=-4k(x-2)$ $y=-4kx+8k$

(iv)	 When $y=1$, $1=-4kx+8k \Rightarrow x=\frac{8k-1}{4k}$ $\therefore$ required area $=\frac{1}{2}\left(\frac{8k-1}{4k}+2\right)(1)$ $=\frac{16k-1}{8k}$ $=2-\frac{1}{8k}$ $>2-\frac{1}{8}$ (since $k>1$) $>\frac{15}{8}$
8 (i)	Area $=2\int_0^{\pi} x^2 \sin x \,\mathrm{d}x$ $=2\left[\left[-x^2\cos x\right]_0^{\pi}+\int_0^{\pi} 2x\cos x\,\mathrm{d}x\right]$ $=2\left[\pi^2+2(\left[x\sin x\right]_0^{\pi}-\int_0^{\pi}\sin x\,\mathrm{d}x)\right]$ $=2\left[\pi^2+2\left[\cos x\right]_0^{\pi}\right]$ $=2(\pi^2-4)$ units^2
(ii)	

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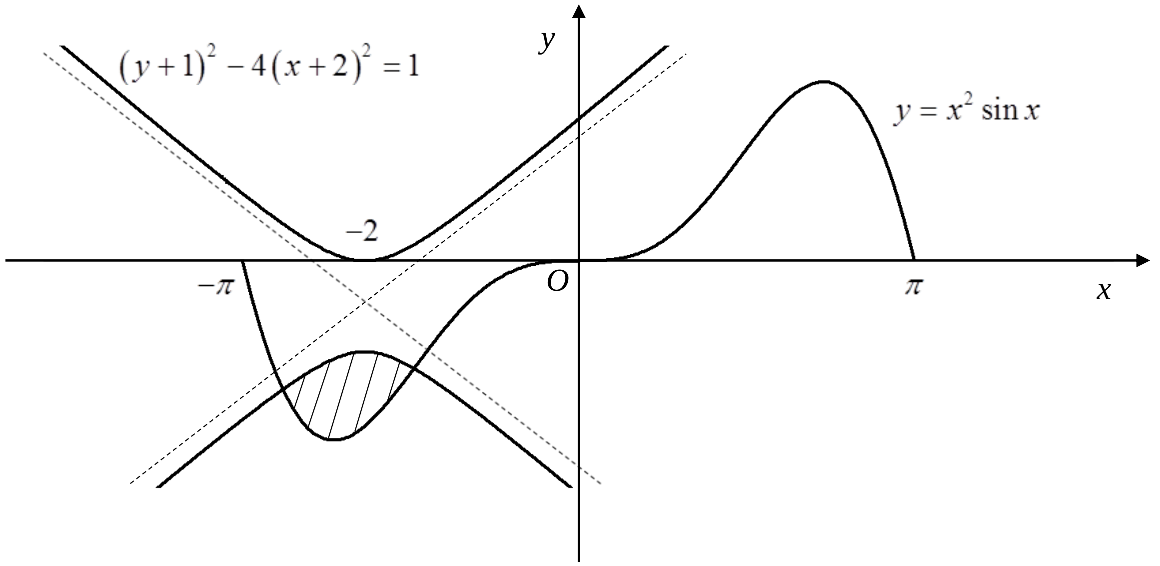


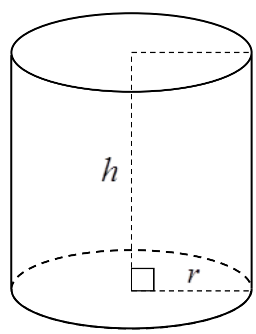
Coordinates of the points of intersections of the 2 curves are $(-1.5374, -2.3623)$ and $(-2.7626, -2.8238)$.

Volume of solid generated

$$= \pi\int_{-2.7626}^{-1.5374} (x^2 \sin x)^2 \, dx - \pi\int_{-2.7626}^{-1.5374} \left(-1-\sqrt{1+4(x+2)^2}\right)^2 dx = 26.8 \text{ units}^3$$

9
(i)

Let A cm^2 be the surface area of the cylindrical container.
Let r cm and h cm be the radius and height of the cylindrical container respectively.



$\text{Volume} = \pi r^2 h = k$

$\therefore \; h = \dfrac{k}{\pi r^2}$

$A = 2\pi r h + \pi r^2$

$\quad = 2\pi r\left(\dfrac{k}{\pi r^2}\right) + \pi r^2$

$\quad = \dfrac{2k}{r} + \pi r^2$

Can also express r in terms of h and find A in terms of h, then let $\dfrac{dA}{dh} = 0$ to obtain h and subsequently r.

Hence $\dfrac{dA}{dr} = -\dfrac{2k}{r^2} + 2\pi r$

When $\dfrac{dA}{dr} = 0$,

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$-\frac{2k}{r^2}+2\pi r=0$

$r^3=\frac{k}{\pi}$

$r=\sqrt[3]{\frac{k}{\pi}}$

$\therefore\ h=\frac{k}{\pi r^2}=\frac{k}{\pi\left[\left(\frac{k}{\pi}\right)^{\frac{1}{3}}\right]^2}=\sqrt[3]{\frac{k}{\pi}}$

Hence $h:r=\sqrt[3]{\frac{k}{\pi}}:\sqrt[3]{\frac{k}{\pi}}=1:1$ (shown)

$\frac{d^2A}{dr^2}=\frac{4k}{r^3}+2\pi>0$ since $p>0$ and $k>0$

Hence A is a minimum when $r=\sqrt[3]{\frac{k}{\pi}}$

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From (i), $h:r=1:1$

Hence $A=2\pi rh+\pi r^2=2\pi r(r)+\pi r^2=3\pi r^2$

For new design, $h:r=5:2$

Hence new $A=2\pi rh+\pi r^2=2\pi r\left(\frac{5}{2}r\right)+\pi r^2=6\pi r^2$

$\therefore$ required ratio is $6\pi r^2:3\pi r^2=2:1$

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Method 1

Let V cm^3 be the volume of the cylindrical container.

$V=\pi r^2h=\pi r^2(\frac{5}{2}r)=\frac{5}{2}\pi r^3$

$A=2\pi rh+\pi r^2=2\pi r\left(\frac{5}{2}r\right)+\pi r^2=6\pi r^2$

$\frac{dV}{dr}=\frac{15}{2}\pi r^2$

Can also find $\frac{dV}{dh}$ and $\frac{dA}{dh}$ and use $\frac{dA}{dt}=\frac{dA}{dh}\times\frac{dh}{dV}\times\frac{dV}{dt}$

$\frac{dA}{dr}=12\pi r$

$\frac{dA}{dt}=\frac{dA}{dr}\times\frac{dr}{dV}\times\frac{dV}{dt}$

$=12\pi r\times\frac{2}{15\pi r^2}\times 80$

$=\frac{128}{r}$

When $h=50$, $r=\frac{2}{5}(50)=20$

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	Hence $\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{128}{20}=6.4\ \text{ cm}^2/\text{s}$ <u>Method 2</u> $A=6\pi r^2 \quad \therefore\ r=\sqrt{\frac{A}{6\pi}}$ (reject $r=-\sqrt{\frac{A}{6\pi}}$ since $r\geq 0$) Hence $V=\pi r^2 h=\pi r^2\left(\frac{5}{2}r\right) \ =\frac{5}{2}\pi r^3$ $=\frac{5}{2}\pi\left(\sqrt{\frac{A}{6\pi}}\right)^3$ $=\frac{5A^{\frac{3}{2}}}{2(6)^{\frac{3}{2}}\pi^{\frac{1}{2}}}$ $\frac{\mathrm{d}V}{\mathrm{d}A}=\frac{15A^{\frac{1}{2}}}{4(6)^{\frac{3}{2}}\pi^{\frac{1}{2}}}$ When $h=50$, $r=\frac{2}{5}(50)=20$ $\therefore\ A=6\pi(20)^2=2400\pi$ Hence $\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{\mathrm{d}A}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}$ $=\frac{4(6)^{\frac{3}{2}}\pi^{\frac{1}{2}}}{15A^{\frac{1}{2}}}\times 80$ $=\frac{4(6)^{\frac{3}{2}}\pi^{\frac{1}{2}}}{15(2400\pi)^{\frac{1}{2}}}\times 80$ $=6.4 \quad \text{cm}^2/\text{s}$
10 (a) (i)	$\frac{\sin\theta}{x}=\frac{\sin\left(\pi-\frac{\pi}{6}-\theta\right)}{y}$ $\frac{x}{y}=\frac{\sin\theta}{\sin\left(\frac{5\pi}{6}-\theta\right)}$ $\frac{x}{y}=\frac{\sin\theta}{\sin\frac{5\pi}{6}\cos\theta-\sin\theta\cos\frac{5\pi}{6}}$ $\frac{x}{y}=\frac{\sin\theta}{\frac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta}=\frac{2\sin\theta}{\cos\theta+\sqrt{3}\sin\theta}$ (shown)

(a) (ii)	$\frac{x}{y}=\frac{2\sin\theta}{\cos\theta+\sqrt{3}\sin\theta}$ $\frac{x}{y}=\frac{2\left(\theta-\frac{\theta^3}{3!}+...\right)}{1+\sqrt{3}\theta-\frac{\theta^2}{2}+...}$ $\frac{x}{y}\approx 2\left(\theta-\frac{\theta^3}{3!}\right)\left(1+\left(\sqrt{3}\theta-\frac{\theta^2}{2}\right)\right)^{-1}$ $\frac{x}{y}\approx 2\left(\theta-\frac{\theta^3}{3!}\right)\left(\begin{array}{l}1+(-1)\left(\sqrt{3}\theta-\frac{\theta^2}{2}\right)\\+\frac{(-1)(-2)}{2!}\left(\sqrt{3}\theta-\frac{\theta^2}{2}\right)^2\end{array}\right)$ $\frac{x}{y}\approx 2\left(\theta-\frac{\theta^3}{3!}\right)\left(1-\sqrt{3}\theta+\frac{\theta^2}{2}+3\theta^2\right)$ $\frac{x}{y}\approx 2\theta-2\sqrt{3}\theta^2+\frac{20}{3}\theta^3$
(b) (i)	Using sine rule, $\frac{\sin\theta}{x}=\frac{\sin\frac{\pi}{6}}{\frac{1}{6}}=3 \quad \therefore \theta=\sin^{-1}3x$
(b) (ii)	<u>Method 1</u> $\sin\theta=3x$ $\cos\theta\frac{d\theta}{dx}=3$ ---- (1) $\cos\theta\frac{d^2\theta}{dx^2}-\sin\theta\left(\frac{d\theta}{dx}\right)^2=0$ --- (2) $\cos\theta\frac{d^3\theta}{dx^3}-\sin\theta\frac{d\theta}{dx}\frac{d^2\theta}{dx^2}-2\sin\theta\frac{d\theta}{dx}\frac{d^2\theta}{dx^2}-\cos\theta\left(\frac{d\theta}{dx}\right)^3$ $=0---(3)$ When $x=0$, $\theta=0,\ \frac{d\theta}{dx}=3,\ \frac{d^2\theta}{dx^2}=0,\ \frac{d^3\theta}{dx^3}=27$ $\theta=3x+\frac{27}{3!}x^3+...=3x+\frac{9}{2}x^3+...$

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<u>Method 2</u>

$\theta = \sin^{-1}(3x)$

$\frac{d\theta}{dx} = \frac{3}{\sqrt{1-9x^2}} = 3\left(1-9x^2\right)^{-\frac{1}{2}}$

$\frac{d^2\theta}{dx^2} = 3\left(-\frac{1}{2}\right)\left(1-9x^2\right)^{-\frac{3}{2}}(-18x) = 27x\left(1-9x^2\right)^{-\frac{3}{2}}$

$\frac{d^3\theta}{dx^3} = -\frac{81}{2}x(-18x)\left(1-9x^2\right)^{-\frac{5}{2}} + 27\left(1-9x^2\right)^{-\frac{3}{2}}$

$= 729x^2\left(1-9x^2\right)^{-\frac{5}{2}} + 27\left(1-9x^2\right)^{-\frac{3}{2}}$

When $x = 0$,

$\theta = 0$, $\frac{d\theta}{dx} = 3$, $\frac{d^2\theta}{dx^2} = 0$, $\frac{d^3\theta}{dx^3} = 27$

$\theta = 3x + \frac{27}{3!}x^3 + \ldots = 3x + \frac{9}{2}x^3 + \ldots$

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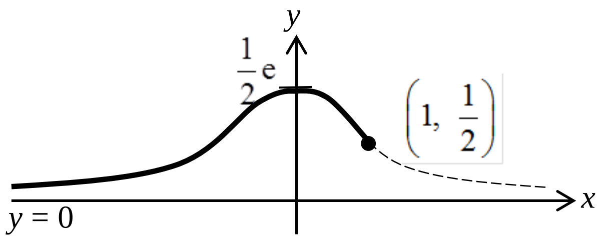
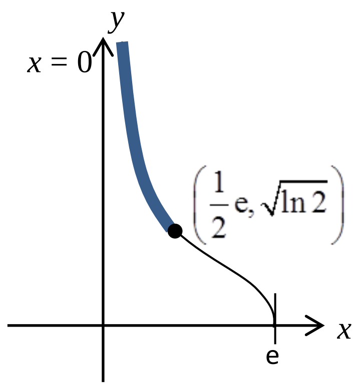
(i)</td><td>



Since $R_f = \left(0, \frac{1}{2}e\right] \subseteq D_g = (0, e]$, $R_f \subseteq D_g$ and gf exists.



$R_{gf} = \left[\sqrt{\ln 2}, \infty\right)$

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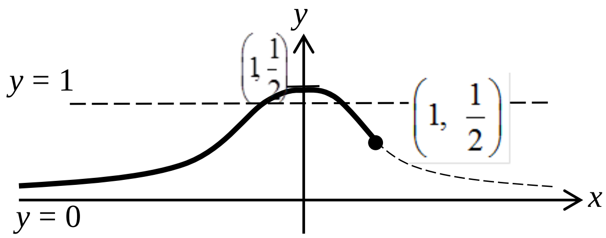
(ii)	 Since a horizontal line y = 1 cuts the graph of y = f(x) twice, f is not a one-to-one function and f^{-1} does not exist.
(iii)	$b=0$ Let $y=\mathrm{f}(x)$ $y=\frac{1}{2}\mathrm{e}^{1-x^2}$ $\ln(2y)=1-x^2$ $x=\pm\sqrt{1-\ln(2y)}$ Since $x\le 0,\ x=-\sqrt{1-\ln(2y)}$ $\mathrm{f}^{-1}:x\mapsto -\sqrt{1-\ln(2x)},\ x\in\mathbb{R},\ 0<x\le\frac{1}{2}\mathrm{e}$
(iv)	$y=\sqrt{1-\ln x}\xrightarrow{\text{Step 1}} y=\sqrt{1-\ln\left(\frac{x}{2}\right)}$ $y=\sqrt{1-\ln\left(\frac{x}{2}\right)}\xrightarrow{\text{Step 2}} y=-\sqrt{1-\ln\left(\frac{x}{2}\right)}$ $0<x\le\mathrm{e}\rightarrow 0<\frac{x}{2}\le\mathrm{e}$ $\therefore 0<x\le 2\mathrm{e}$ $\mathrm{h}:x\mapsto -\sqrt{1-\ln\left(\frac{x}{2}\right)},\ x\in\mathbb{R},\ 0<x\le 2\mathrm{e}$