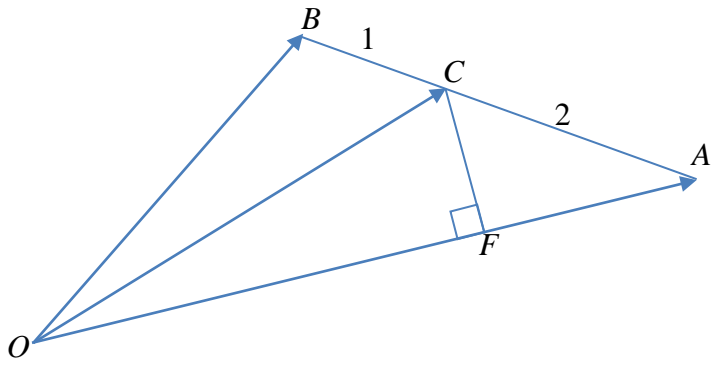


## 2016 SH2 H2 Mathematics Preliminary Examination Paper 1

### Suggested Solutions

Qn No.	Solution
<b>1</b>	<p>Let \$x be the price of a regular cup of coffee. \$y be the price of a medium cup of coffee. \$z be the price of a large cup of coffee.</p> $5x + 3y + 2z = 20.90 \quad (1)$ $3x + 4y + z = 17.10 \quad (2)$ <p>There is a 12.5% discount given to customer C since she has bought more than 12 cups. Hence</p> $2x + 8y + 4z = \frac{100}{87.5} \times 28 = 32 \quad (3)$ <p>Using GC, <math>x = 1.80</math>, <math>y = 2.30</math>, <math>z = 2.50</math></p>

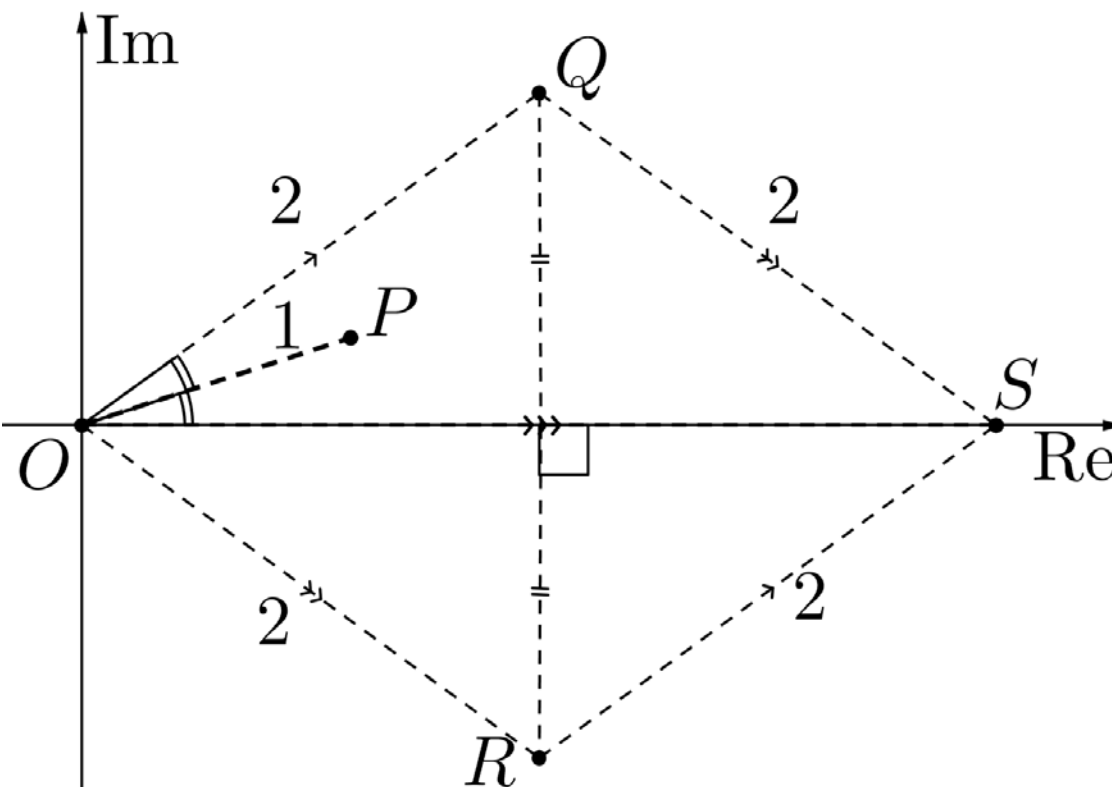
Qn No.	Solution
<b>2 (i)</b>	$\frac{3x^2 + 14}{(x+1)(x+2)} \geq 2$ $\Rightarrow \frac{3x^2 + 14}{(x+1)(x+2)} - 2 \geq 0, \quad x \neq -1, -2$ $\Rightarrow \frac{3x^2 + 14 - (2x^2 + 6x + 4)}{(x+1)(x+2)} \geq 0$ $\Rightarrow \frac{x^2 - 6x + 10}{(x+1)(x+2)} \geq 0$ $\Rightarrow (x+1)(x+2)(x^2 - 6x + 10) \geq 0, \quad x \neq -1, -2.$ <p><b>Method 1: Completing the square</b></p> $\Rightarrow (x+1)(x+2)\left[(x-3)^2 + 1\right] \geq 0, \quad x \neq -1, -2.$ <p>Since <math>(x-3)^2 + 1 \geq 0</math> for all values of <math>x</math>,</p> $\Rightarrow (x+1)(x+2) \geq 0 \text{ and } x \neq -1, -2.$ $\Rightarrow x < -2 \text{ or } x > -1.$ <p><b>Method 2: Using both the discriminant and coefficient of <math>x^2</math></b></p> <p>For <math>x^2 - 6x + 10</math>, discriminant <math>= 6^2 - 4(1)(10) = -4 &lt; 0</math> and the coefficient of <math>x^2</math> is positive. Hence the graph of <math>y = x^2 - 6x + 10</math> lies entirely above the <math>x</math>-axis, which implies that <math>x^2 - 6x + 10 \geq 0</math> for all values of <math>x</math>.</p>
<b>2 (ii)</b>	$\frac{3x^2 + 14}{( x -1)( x -2)} \geq 2$ $\frac{3(- x )^2 + 14}{(- x +1)(- x +2)} \geq 2$ <p>Therefore <math>- x  &lt; -2</math> or <math>- x  &gt; -1</math></p> $\Rightarrow  x  > 2 \text{ or }  x  < 1$ $\Rightarrow x < -2 \text{ or } -1 < x < 1 \text{ or } x > 2$

Qn No.	Solution
3	 <p>By ratio theorem, <math>\overrightarrow{OC} = \frac{2\mathbf{b} + \mathbf{a}}{3}</math>.</p>
	$\mathbf{a} \cdot \mathbf{b} =  \mathbf{a}   \mathbf{b}  \cos 60^\circ = \frac{1}{2}  \mathbf{a}   \mathbf{b} $ <p>Length of projection of <math>\overrightarrow{OC}</math> on <math>\overrightarrow{OA}</math></p> $OF = \left  \overrightarrow{OC} \cdot \frac{\overrightarrow{OA}}{ \overrightarrow{OA} } \right $ $= \frac{1}{3} \left  \frac{(2\mathbf{b} + \mathbf{a}) \cdot \mathbf{a}}{ \mathbf{a} } \right $ $= \frac{1}{3} \left  \frac{2\mathbf{b} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{a}}{ \mathbf{a} } \right $ $= \frac{1}{3} \left  \frac{2\mathbf{b} \cdot \mathbf{a} +  \mathbf{a} ^2}{ \mathbf{a} } \right $ $= \frac{1}{3} \left( \frac{2 \mathbf{a} \cdot \mathbf{b} }{ \mathbf{a} } +  \mathbf{a}  \right)$ $= \frac{1}{3} \left( \frac{2 \mathbf{a}   \mathbf{b}  \cos 60^\circ}{ \mathbf{a} } +  \mathbf{a}  \right)$ $= \frac{1}{3} \left( \frac{2 \mathbf{a}   \mathbf{b} }{2 \mathbf{a} } +  \mathbf{a}  \right)$ $= \frac{1}{3} ( \mathbf{a}  +  \mathbf{b} )$

Qn No.	Solution
<b>4 (a)</b>	<p><b>Method 1A</b></p> $2w - z = 6i \quad \dots\dots(1)$ $wz = \frac{13}{2} \quad \dots\dots(2) \Rightarrow w = \frac{13}{2z} \quad \dots\dots(3)$ <p>Substituting (3) into (1), <math>2\left(\frac{13}{2z}\right) - z = 6i</math></p> $\frac{13}{z} - z = 6i$ $13 - z^2 = 6iz$ $z^2 + 6iz - 13 = 0$ $z^2 + 6iz + (3i)^2 - (3i)^2 - 13 = 0$ $(z + 3i)^2 = 4$ $z = 2 - 3i \text{ or } -2 - 3i$ <p>When <math>z = 2 - 3i</math>, <math>w = \frac{13}{2(2 - 3i)} = 1 + \frac{3}{2}i</math></p> <p>When <math>z = -2 - 3i</math>, <math>w = \frac{13}{2(-2 - 3i)} = -1 + \frac{3}{2}i</math></p> <p><math>\therefore w = 1 + \frac{3}{2}i, z = 2 - 3i</math> or <math>w = -1 + \frac{3}{2}i, z = -2 - 3i</math></p> <p><b>Method 1B</b></p> $2w - z = 6i \quad \dots\dots(1)$ $wz = \frac{13}{2} \quad \dots\dots(2) \Rightarrow z = \frac{13}{2w} \quad \dots\dots(3)$ <p>Substituting (3) into (1), <math>2w - \frac{13}{2w} = 6i</math></p> $4w^2 - 13 = 12iw$ $4w^2 - 12iw - 13 = 0$ $4\left(w^2 - 3iw - \frac{9}{4}\right) + 9 - 13 = 0$ $4\left(w - \frac{3i}{2}\right)^2 - 4 = 0$ $\left(w - \frac{3i}{2}\right)^2 = 1 \Rightarrow w = 1 + \frac{3i}{2} \text{ or } -1 + \frac{3i}{2}$ <p>When <math>w = 1 + \frac{3}{2}i</math>, <math>z = \frac{13}{2\left(1 + \frac{3}{2}i\right)} = 2 - 3i</math></p> <p>When <math>w = -1 + \frac{3}{2}i</math>, <math>z = \frac{13}{2\left(-1 + \frac{3}{2}i\right)} = -2 - 3i</math></p> <p><math>\therefore w = 1 + \frac{3}{2}i, z = 2 - 3i</math> or <math>w = -1 + \frac{3}{2}i, z = -2 - 3i</math></p>

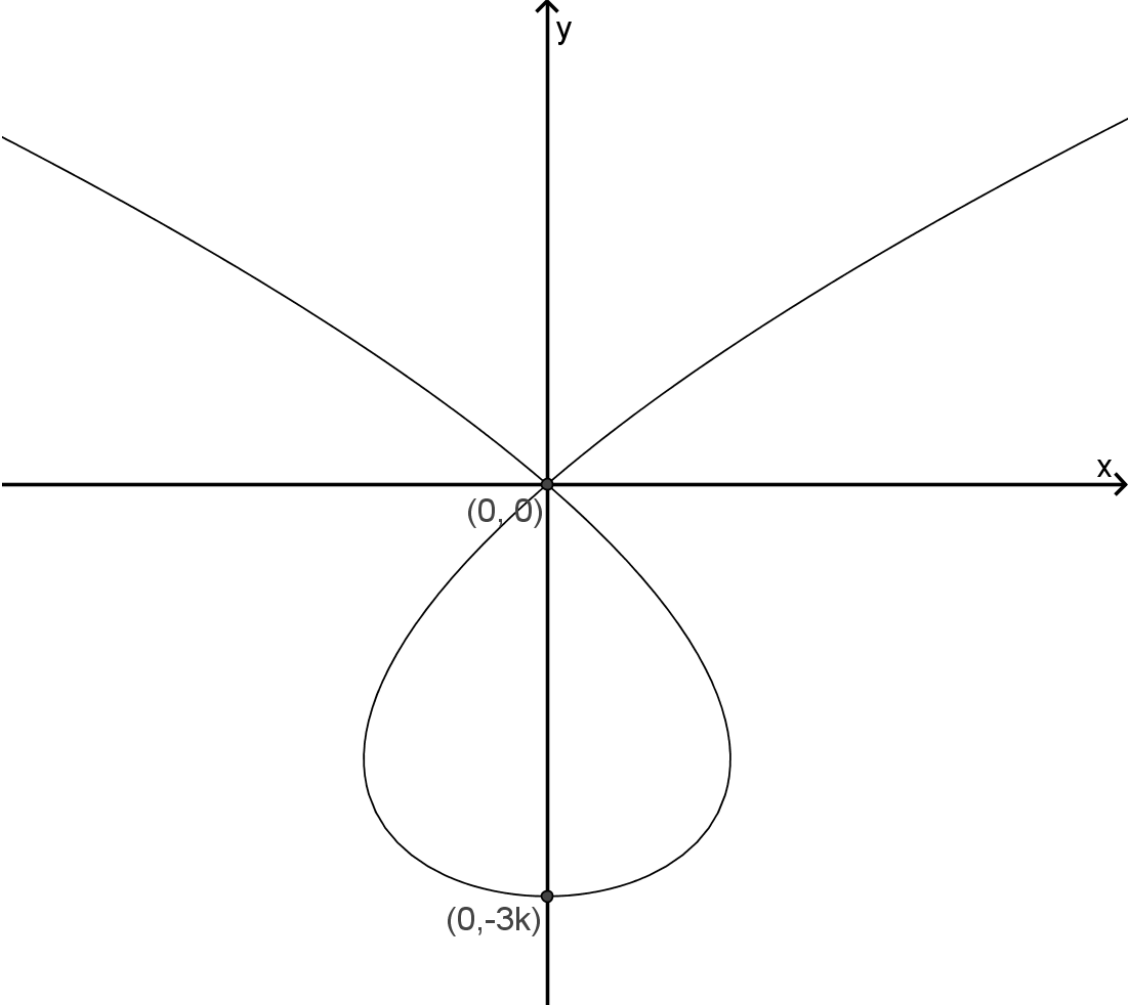
Qn No.	Solution
<b>4 (a)</b>	<p><b>Method 2A</b></p> $2w - z = 6i \quad \dots\dots(1) \Rightarrow z = 6i - 2w \quad \dots\dots(3)$ $wz = \frac{13}{2} \quad \dots\dots(2)$ <p>Substituting (3) into (2), <math>w(6i - 2w) = \frac{13}{2}</math></p> $w(6i - 2w) = \frac{13}{2}$ $4w^2 - 12iw = 13$ $4w^2 - 12iw - 13 = 0$ $4\left(w^2 - 3iw - \frac{9}{4}\right) + 9 - 13 = 0$ $4\left(w - \frac{3i}{2}\right)^2 - 4 = 0$ $\left(w - \frac{3i}{2}\right)^2 = 1 \Rightarrow w = 1 + \frac{3i}{2} \text{ or } -1 + \frac{3i}{2}$ <p>When <math>w = 1 + \frac{3i}{2}</math>, <math>z = \frac{13}{2\left(1 + \frac{3i}{2}\right)} = 2 - 3i</math></p> <p>When <math>w = -1 + \frac{3i}{2}</math>, <math>z = \frac{13}{2\left(-1 + \frac{3i}{2}\right)} = -2 - 3i</math></p> <p><math>\therefore w = 1 + \frac{3i}{2}</math>, <math>z = 2 - 3i</math> or <math>w = -1 + \frac{3i}{2}</math>, <math>z = -2 - 3i</math></p> <p><b>Method 2B</b></p> $2w - z = 6i \quad \dots\dots(1) \Rightarrow 2w = 6i + z \quad \dots\dots(3)$ $wz = \frac{13}{2} \quad \dots\dots(2)$ <p>Substituting (3) into (2), <math>(6i + z)z = 13</math></p> $6iz + z^2 = 13$ $z^2 + 6iz - 13 = 0$ $z^2 + 6iz + (3i)^2 - (3i)^2 - 13 = 0$ $(z + 3i)^2 = 4$ $z = 2 - 3i \text{ or } -2 - 3i$ <p>When <math>z = 2 - 3i</math>, <math>w = \frac{13}{2(2 - 3i)} = 1 + \frac{3i}{2}</math></p> <p>When <math>z = -2 - 3i</math>, <math>w = \frac{13}{2(-2 - 3i)} = -1 + \frac{3i}{2}</math></p> <p><math>\therefore w = 1 + \frac{3i}{2}</math>, <math>z = 2 - 3i</math> or <math>w = -1 + \frac{3i}{2}</math>, <math>z = -2 - 3i</math></p>

Qn No.	Solution
<b>4 (a)</b>	<p><b>Method 3</b></p> $2w - z = 6i \quad \dots\dots(1)$ $wz = \frac{13}{2} \quad \dots\dots(2)$ $(2) \Rightarrow w = \frac{13}{2z} \quad \dots\dots(3)$ <p>Substituting (3) into (1),</p> $2\left(\frac{13}{2z}\right) - z = 6i$ $\frac{13}{z} - z = 6i$ $13 - z^2 = 6iz$ $z^2 + 6iz - 13 = 0$ <p>Let <math>z = x + yi</math>. Then</p> $(x + iy)^2 + 6i(x + iy) - 13 = 0$ $x^2 - y^2 + 2ixy + 6ix - 6y - 13 = 0$ $x^2 - y^2 - 6y - 13 + i(2xy + 6x) = 0$ <p>Comparing imaginary parts,</p> $2xy + 6x = 0$ $x(y + 3) = 0$ $x = 0 \text{ or } y = -3$ <p>If <math>x = 0</math>, comparing real parts,</p> $-y^2 - 6y - 13 = 0$ $y^2 + 6y + 13 = 0$ $(y + 3)^2 + 4 = 0$ <p>Therefore, there is no solution if <math>x = 0</math>.</p> <p>If <math>y = -3</math>, comparing real parts,</p> $x^2 - 3^2 - 6(-3) - 13 = 0$ $x^2 - 4 = 0$ $x = \pm 2$ <p>When <math>x = 2</math>, <math>z = 2 - 3i</math>, <math>w = \frac{13}{2(2 - 3i)} = 1 + \frac{3}{2}i</math></p> <p>When <math>x = -2</math>, <math>z = -2 - 3i</math>, <math>w = \frac{13}{2(-2 - 3i)} = -1 + \frac{3}{2}i</math></p> <p><math>\therefore w = 1 + \frac{3}{2}i</math>, <math>z = 2 - 3i</math> or <math>w = -1 + \frac{3}{2}i</math>, <math>z = -2 - 3i</math></p>

Qn No.	Solution
4 (b)	 <p>Let <math>p = e^{i\alpha}</math>. Then <math>q = 2e^{i(2\alpha)}</math>.</p> <p>Let <math>S</math> represent</p> $q^* + 2p^2 = 2e^{i(-2\alpha)} + 2e^{i(2\alpha)} = 2\operatorname{Re}(e^{i(2\alpha)}) = 2\cos 2\alpha,$ <p>i.e. <math>S</math> represents the point whose real coordinate is twice that of <math>Q</math>, and imaginary coordinate zero.</p>

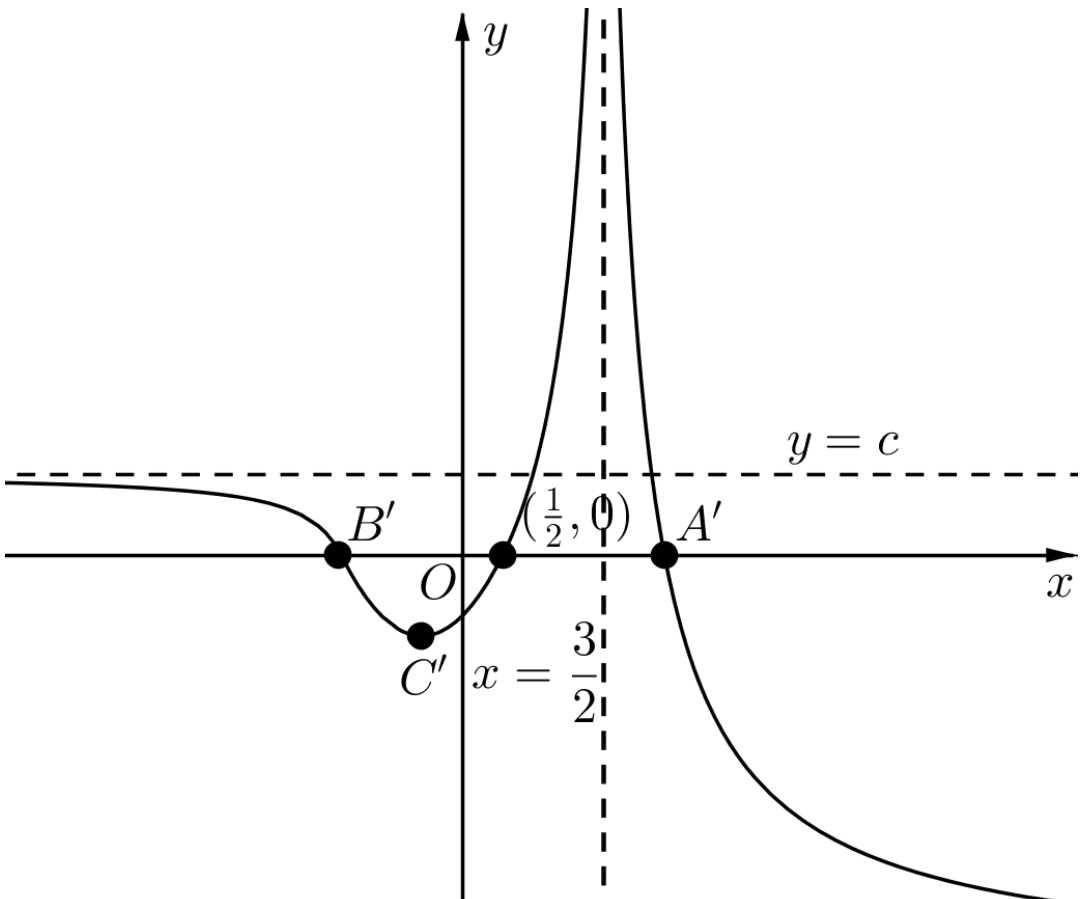
Qn No.	Solution
5	$V = x^2h + \frac{1}{3}x^2\left(\frac{1}{2}x\right) = x^2\left(h + \frac{1}{6}x\right)$ $h = \frac{V}{x^2} - \frac{1}{6}x$ $S = 4xh + x^2 + 4\left(\frac{1}{2}\right)x\left(\frac{1}{\sqrt{2}}x\right)$ $= 4xh + (1 + \sqrt{2})x^2$ <p>Substituting <math>h = \frac{V}{x^2} - \frac{1}{6}x</math>:</p> $S = 4x\left(\frac{V}{x^2} - \frac{1}{6}x\right) + (1 + \sqrt{2})x^2$ $= \frac{4V}{x} + \left(\frac{1}{3} + \sqrt{2}\right)x^2$ <p>For minimum surface area:</p> $\frac{dS}{dx} = -\frac{4V}{x^2} + 2\left(\frac{1}{3} + \sqrt{2}\right)x = 0.$ $\frac{4V}{x^2} = 2\left(\frac{1}{3} + \sqrt{2}\right)x$ $2V = \left(\frac{1}{3} + \sqrt{2}\right)x^3$ $6V = (1 + 3\sqrt{2})x^3$ $x^3 = \frac{6V}{1 + 3\sqrt{2}}$ $x = \left(\frac{6V}{1 + 3\sqrt{2}}\right)^{\frac{1}{3}} = 1.0460049(V^{\frac{1}{3}})$ $= 1.046(V^{\frac{1}{3}}) \text{ (3d.p)}$ <p>Min <math>S = A</math></p> $= \frac{4V}{1.0460049(V^{\frac{1}{3}})} + \left(\frac{1}{3} + \sqrt{2}\right)\left(1.0460049(V^{\frac{1}{3}})\right)^2$ $= \left(\frac{4}{1.0460049} + \left(\frac{1}{3} + \sqrt{2}\right)(1.0460049)^2\right)(V^{\frac{2}{3}})$ $= 5.736110(V^{\frac{2}{3}})$ $= 5.736(V^{\frac{2}{3}}) \text{ (3d.p)}$ <p>Remark: 1<sup>st</sup> or 2<sup>nd</sup> derivative test is not necessary as it is given in the question that the surface area had a minimum value.</p>

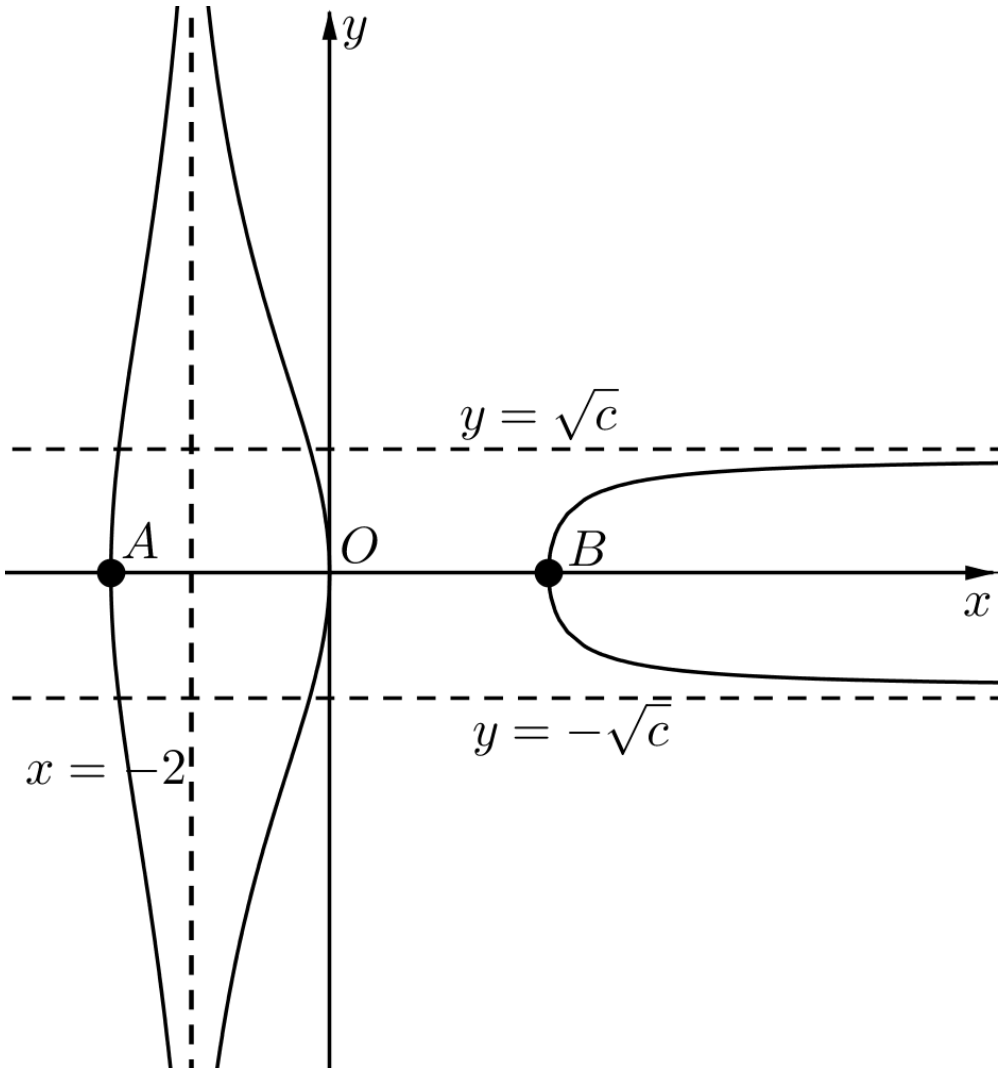
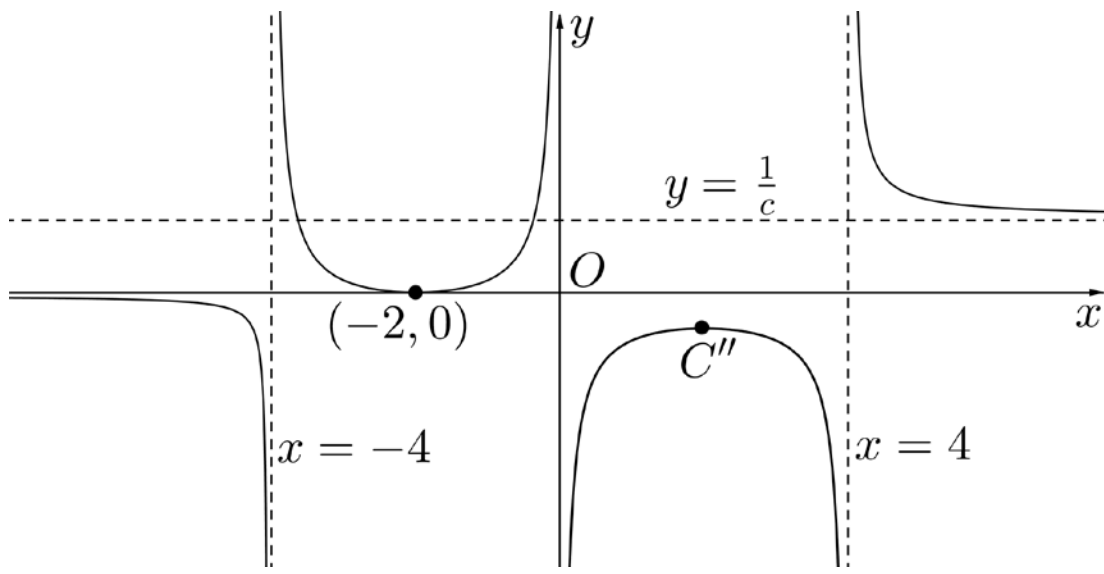


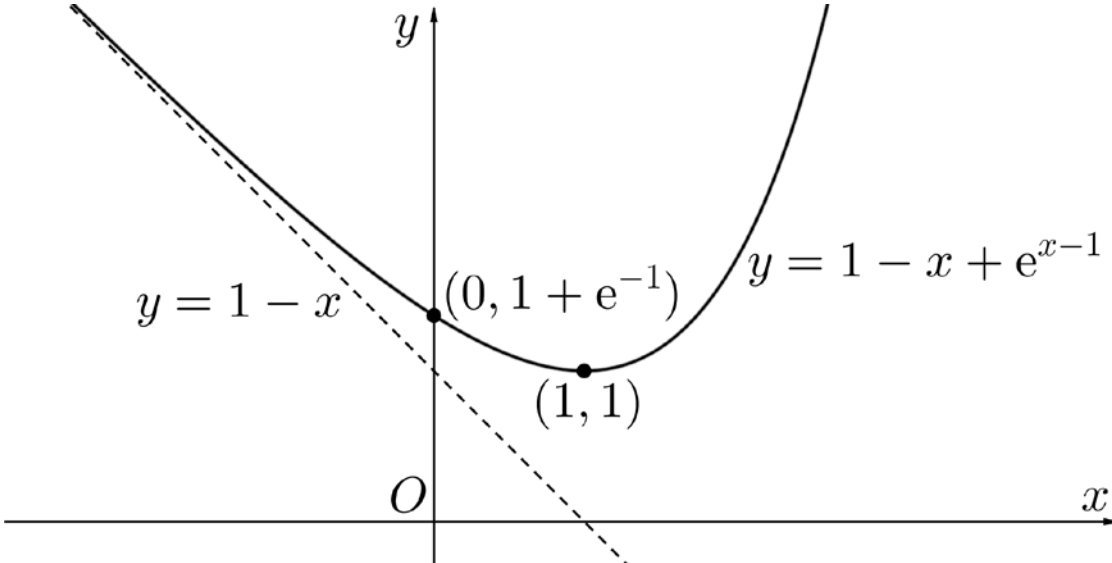
Qn No.	Solution
6 (i)	<div style="text-align: center;">  </div> <div style="display: flex; justify-content: space-around; margin-top: 20px;"> <div style="width: 45%;"> <p>To find y-intercept, let <math>x = 0</math>:</p> <math display="block">t^3 - kt = 0</math> <math display="block">t(t^2 - k) = 0</math> <math display="block">t = 0 \quad \text{or} \quad t = \pm\sqrt{k}</math> <math display="block">y = -3k \quad \quad y = 0</math> <math display="block">\therefore (0, -3k) \quad (0, 0)</math> </div> <div style="width: 45%; border-left: 1px solid black; padding-left: 10px;"> <p>To find x-intercept, let <math>y = 0</math>:</p> <math display="block">3(t^2 - k) = 0</math> <math display="block">t = \pm\sqrt{k}</math> <math display="block">x = 0</math> <math display="block">\therefore (0, 0)</math> </div> </div>
6 (ii)	$x = t^3 - kt \Rightarrow \frac{dx}{dt} = 3t^2 - k$ $y = 3(t^2 - k) \Rightarrow \frac{dy}{dt} = 6t$ $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6t}{3t^2 - k}$

Qn No.	Solution
<b>6 (iii)</b>	<p>When <math>t = -\sqrt{\frac{k}{3}}</math>,</p> $\frac{dy}{dx} = \frac{6t}{3t^2 - k} \text{ is undefined since } 3t^2 - k = 3\left(\frac{k}{3}\right) - k = 0.$ <p>Hence, the tangent is a vertical line with equation</p> $\begin{aligned} x &= t^3 - kt \\ &= t(t^2 - k) \\ &= \left(-\sqrt{\frac{k}{3}}\right)\left(\frac{k}{3} - k\right) \\ &= \frac{2}{3}k\sqrt{\frac{k}{3}} \end{aligned}$
<b>6 (iv)</b>	<p>The tangent passes through the point <math>\left(\frac{2}{3}k, k\right)</math>, therefore</p> $x = \frac{2}{3}k\sqrt{\frac{k}{3}}$ $\frac{2}{3}k = \frac{2}{3}k\sqrt{\frac{k}{3}}$ $k\left(1 - \sqrt{\frac{k}{3}}\right) = 0$ $k = 0 \quad \text{or} \quad 1 - \sqrt{\frac{k}{3}} = 0$ <p>(NA since <math>k &gt; 0</math>)</p> $\frac{k}{3} = 1$ $k = 3$

Qn No.	Solution
<b>7 (i)</b>	$y = \ln(\sec x)$ $\frac{dy}{dx} = \frac{1}{\sec x}(\sec x \tan x) = \tan x$ $\frac{d^2y}{dx^2} = \sec^2 x$ $\frac{d^3y}{dx^3} = 2 \sec x(\sec x \tan x)$ $\frac{d^3y}{dx^3} = 2(\sec^2 x)(\tan x) = 2\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right) \text{ (shown)}$
<b>7 (ii)</b>	$\frac{d^4y}{dx^4} = 2\left(\frac{d^3y}{dx^3}\right)\left(\frac{dy}{dx}\right) + 2\left(\frac{d^2y}{dx^2}\right)^2$ <p>When <math>x = 0</math>,</p> $y = \ln(\sec 0) = \ln 1 = 0 \qquad \frac{dy}{dx} = \tan 0 = 0$ $\frac{d^2y}{dx^2} = \sec^2 0 = 1 \qquad \frac{d^3y}{dx^3} = 2(1)(0) = 0$ $\frac{d^4y}{dx^4} = 2(0)(0) + 2(1)^2 = 2$ <p>Thus, Maclaurin series for <math>y</math> is</p> $y \approx +0x + 1\left(\frac{x^2}{2!}\right) + 0x^3 + 2\left(\frac{x^4}{4!}\right)$ $= \frac{1}{2}x^2 + \frac{1}{12}x^4$
<b>7 (iii)</b>	$\frac{1}{12}x^2 + \ln(\sec x) = \cos 2x$ <p>Using the above results and the first three terms of Maclaurin Series of <math>\cos 2x</math>,</p> $\frac{1}{12}x^2 + \frac{1}{2}x^2 + \frac{1}{12}x^4 \approx 1 - \frac{1}{2!}(2x)^2 + \frac{1}{4!}(2x)^4$ $\frac{7}{12}x^2 + \frac{1}{12}x^4 = 1 - 2x^2 + \frac{2}{3}x^4$ $\frac{7}{12}x^4 - \frac{31}{12}x^2 + 1 = 0$ $7x^4 - 31x^2 + 12 = 0$ $(7x^2 - 3)(x^2 - 4) = 0$ $x^2 = \frac{3}{7} \text{ or } x^2 = 4$ $x = \pm\sqrt{\frac{3}{7}} \text{ or } \pm 2$ <p>Since <math>\alpha</math> is positive and close to zero,</p> $\alpha = \sqrt{\frac{3}{7}}$

Qn No.	Solution
8 (a)	 <p>Coordinates of <math>A'</math>, <math>B'</math> and <math>C'</math> are <math>\left(\frac{5}{2}, 0\right)</math>, <math>\left(-\frac{3}{2}, 0\right)</math> and <math>\left(\frac{1-a}{2}, \frac{b}{2}\right)</math></p>

Qn No.	Solution
8 (b)	 <p>The graph shows a curve in the Cartesian plane. A vertical dashed line represents a line of symmetry at <math>x = -2</math>. A horizontal dashed line represents a horizontal asymptote at <math>y = \sqrt{c}</math>. The curve passes through the origin <math>O</math> and a point <math>A</math> on the x-axis. A point <math>B</math> is marked on the x-axis to the right of the origin. The curve has a cusp-like shape near <math>x = -2</math> and approaches the horizontal asymptote as <math>x \rightarrow \infty</math>.</p>
8 (c)	 <p>The graph shows a function with two vertical asymptotes at <math>x = -4</math> and <math>x = 4</math>, and a horizontal asymptote at <math>y = \frac{1}{c}</math>. The curve has a local minimum at <math>(-2, 0)</math> and a local maximum at <math>C''</math>.</p> <p>Coordinates of <math>C''</math> are <math>\left(a, -\frac{2}{b}\right)</math>.</p>

Qn No.	Solution
9 (i)	$y = z - x \Rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 1$ <p>The <math>\frac{1}{y} \frac{dy}{dx} - 1 = \frac{x-2}{y}</math> becomes</p> $\frac{1}{z-x} \left( \frac{dz}{dx} - 1 \right) - 1 = \frac{x-2}{z-x} \Rightarrow \frac{dz}{dx} - 1 - (z-x) = x-2$ $\Rightarrow \frac{dz}{dx} = z - 1$ $\Rightarrow \int \frac{1}{z-1} dz = \int 1 dx$ $\Rightarrow \ln z-1  = x + c$ $\Rightarrow  z-1  = e^{x+c} = Ae^x$ $\Rightarrow z-1 = \pm Ae^x = Be^x$ $\Rightarrow y = 1 - x + Be^x$ <p>Since the solution curve passes through (1,1), then <math>1 = 1 - 1 + Be \Rightarrow B = e^{-1}</math>. Hence the equation of curve is <math>y = 1 - x + e^{x-1}</math>.</p>
9 (ii)	 <p>The graph shows a Cartesian coordinate system with x and y axes. A dashed line represents the tangent line <math>y = 1 - x</math>. A solid curve represents the solution <math>y = 1 - x + e^{x-1}</math>. The curve is concave up and passes through the point <math>(1, 1)</math>, which is the point of tangency. Another point on the curve is labeled <math>(0, 1 + e^{-1})</math>. The origin is labeled <math>O</math>.</p>

Qn No.	Solution
10 (a)	$\frac{5x^2 - 2x + 7}{(1-x)(2x^2 + 3)} = \frac{A}{1-x} + \frac{Bx + C}{2x^2 + 3}, \text{ where}$ $A = \frac{5 - 2 + 7}{2 + 3} = 2,$ $5x^2 - 2x + 7 = 2(2x^2 + 3) + (Bx + C)(1 - x)$ <p>Comparing the coefficients of <math>x^2</math> and <math>x^0</math>, we have</p> $4 - B = 5 \Rightarrow B = -1, \text{ and}$ $6 + C = 7 \Rightarrow C = 1.$ <p>Hence we get</p> $\frac{5x^2 - 2x + 7}{(1-x)(2x^2 + 3)} = \frac{2}{1-x} + \frac{-x+1}{2x^2 + 3}$ <p>Therefore,</p> $\int \frac{5x^2 - 2x + 7}{(1-x)(2x^2 + 3)} dx$ $= \int \frac{2}{1-x} + \frac{-x+1}{2x^2 + 3} dx$ $= \int \frac{2}{1-x} dx + \int \frac{-x+1}{2x^2 + 3} dx$ $= \int \frac{2}{1-x} dx - \int \frac{x}{2x^2 + 3} dx + \int \frac{1}{2x^2 + 3} dx$ $= -2 \int \frac{-1}{1-x} dx - \frac{1}{4} \int \frac{4x}{2x^2 + 3} dx + \frac{1}{2} \int \frac{1}{x^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dx$ $= -2 \ln 1-x  - \frac{1}{4} \ln 2x^2 + 3  + \frac{1}{2} \cdot \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1} \left( \frac{x}{\frac{\sqrt{3}}{2}} \right) + c$ $= -2 \ln 1-x  - \frac{1}{4} \ln 2x^2 + 3  + \frac{1}{\sqrt{6}} \tan^{-1} \left( \sqrt{\frac{2}{3}} x \right) + c$

Qn No.	Solution
10 (b) (i)	$\frac{d}{dx} [\sin(e^{-x})] = -e^{-x} \cos(e^{-x})$
10 (b) (ii)	$\begin{aligned} & \int_0^n e^{-2x} \cos(e^{-x}) dx \\ &= \int_0^n e^{-x} (e^{-x} \cos(e^{-x})) dx \\ &= [e^{-x} (-\sin(e^{-x}))]_0^n - \int_0^n (-\sin(e^{-x})) (-e^{-x}) dx \\ &= [e^{-n} (-\sin(e^{-n}))] - [e^0 (-\sin(e^0))] + \int_0^n -e^{-x} \sin(e^{-x}) dx \\ &= -e^{-n} \sin(e^{-n}) + \sin 1 + [-\cos(e^{-x})]_0^n \\ &= -e^{-n} \sin(e^{-n}) + \sin 1 + [-\cos(e^{-n}) + \cos(e^0)] \\ &= -e^{-n} \sin(e^{-n}) - \cos(e^{-n}) + \sin 1 + \cos 1 \end{aligned}$
10 (b) (iii)	$\begin{aligned} & \int_0^\infty e^{-2x} \cos(e^{-x}) dx \\ &= \lim_{n \rightarrow \infty} \int_0^n e^{-2x} \cos(e^{-x}) dx \\ &= \lim_{n \rightarrow \infty} [-e^{-n} \sin(e^{-n}) - \cos(e^{-n}) + \sin 1 + \cos 1] \\ &= \sin 1 + \cos 1 - 1 \end{aligned}$



Qn No.	Solution
11 (a)	<p>Let <math>P_n</math> be the statement</p> $\frac{2}{1^2 \times 3^2} + \frac{3}{2^2 \times 4^2} + \dots + \frac{n+1}{n^2(n+2)^2} = \frac{5}{16} - \frac{1}{4(n+1)^2} - \frac{1}{4(n+2)^2}$ <p>for all <math>n \in \mathbb{Z}^+</math>.</p> <p>When <math>n = 1</math>,</p> $\text{LHS} = \frac{2}{1^2 \times 3^2} = \frac{2}{9}$ $\text{RHS} = \frac{5}{16} - \frac{1}{4(1+1)^2} - \frac{1}{4(1+2)^2}$ $= \frac{5}{16} - \frac{1}{16} - \frac{1}{36} = \frac{2}{9} = \text{LHS}$ <p>Thus, <math>P_1</math> is true.</p> <p>Assume <math>P_k</math> is true for some <math>k \in \mathbb{Z}^+</math>, i.e.</p> $\frac{2}{1^2 \times 3^2} + \frac{3}{2^2 \times 4^2} + \dots + \frac{k+1}{k^2(k+2)^2} = \frac{5}{16} - \frac{1}{4(k+1)^2} - \frac{1}{4(k+2)^2}$ <p>Consider <math>P_{k+1}</math>: To show</p> $\frac{2}{1^2 \times 3^2} + \frac{3}{2^2 \times 4^2} + \dots + \frac{k+2}{(k+1)^2(k+3)^2} = \frac{5}{16} - \frac{1}{4(k+2)^2} - \frac{1}{4(k+3)^2}$ $\text{LHS of } P_{k+1} = \frac{2}{1^2 \times 3^2} + \frac{3}{2^2 \times 4^2} + \dots + \frac{k+1}{k^2(k+2)^2} + \frac{k+2}{(k+1)^2(k+3)^2}$ $= \frac{5}{16} - \frac{1}{4(k+1)^2} - \frac{1}{4(k+2)^2} + \frac{k+2}{(k+1)^2(k+3)^2}$ $= \frac{5}{16} - \frac{1}{4(k+2)^2} + \frac{k+2}{(k+1)^2(k+3)^2} - \frac{1}{4(k+1)^2}$ $= \frac{5}{16} - \frac{1}{4(k+2)^2} + \frac{4(k+2) - (k+3)^2}{4(k+1)^2(k+3)^2}$ $= \frac{5}{16} - \frac{1}{4(k+2)^2} + \frac{4k+8 - k^2 - 6k - 9}{4(k+1)^2(k+3)^2}$ $= \frac{5}{16} - \frac{1}{4(k+2)^2} + \frac{-k^2 - 2k - 1}{4(k+1)^2(k+3)^2}$ $= \frac{5}{16} - \frac{1}{4(k+2)^2} + \frac{-(k+1)^2}{4(k+1)^2(k+3)^2}$ $= \frac{5}{16} - \frac{1}{4(k+2)^2} - \frac{1}{4(k+3)^2}$ $= \text{RHS of } P_{k+1}$ <p><math>P_k</math> is true <math>\Rightarrow P_{k+1}</math> is true.</p> <p>Since <math>P_1</math> is true, and <math>P_k</math> is true <math>\Rightarrow P_{k+1}</math> is true, by mathematical induction, <math>P_n</math> is true for all <math>n \in \mathbb{Z}^+</math>.</p>

Qn No.	Solution
11 (b) (i)	$\frac{4n+5}{n(n+1)} = \frac{5}{n} - \frac{1}{n+1}$ $\sum_{n=1}^N \frac{4n+5}{n(n+1)} \left( \frac{1}{5^{n+1}} \right) = \sum_{n=1}^N \left( \frac{5}{n} - \frac{1}{n+1} \right) \left( \frac{1}{5^{n+1}} \right)$ $= \sum_{n=1}^N \left( \frac{1}{n5^n} - \frac{1}{(n+1)5^{n+1}} \right)$ $= \frac{1}{1 \cdot 5^1} - \frac{1}{2 \cdot 5^2}$ $+ \frac{1}{2 \cdot 5^2} - \frac{1}{3 \cdot 5^3}$ $= \dots$ $+ \frac{1}{(N-1)5^{N-1}} - \frac{1}{N5^N}$ $+ \frac{1}{N5^N} - \frac{1}{(N+1)5^{N+1}}$ $= \frac{1}{5} - \frac{1}{(N+1)5^{N+1}} \text{ (shown)}$ <p><math>\therefore a = \frac{1}{5}, b = -1</math></p>
11 (b) (ii)	$\sum_{n=1}^{\infty} \frac{4n+5}{n(n+1)} \left( \frac{1}{5^{n+1}} \right) = \lim_{N \rightarrow \infty} \left[ \frac{1}{5} - \frac{1}{(N+1)5^{N+1}} \right] = \frac{1}{5}$
11 (b) (iii)	<p><b>Method 1: Substitution of Indices</b></p> $\sum_{n=2}^{N-2} \left[ \frac{4n+1}{n(n-1)} \left( \frac{1}{5^n} \right) \right] = \sum_{n+1=2}^{n+1=N-2} \left[ \frac{4(n+1)+1}{(n+1)((n+1)-1)} \left( \frac{1}{5^{n+1}} \right) \right]$ $= \sum_{n=1}^{N-3} \left[ \frac{4n+5}{n(n+1)} \left( \frac{1}{5^{n+1}} \right) \right]$ $= \frac{1}{5} - \frac{1}{(N-2)5^{N-2}}$ <p><b>Method 2: Listing of Terms</b></p> $\sum_{n=2}^{N-2} \left[ \frac{4n+1}{n(n-1)} \left( \frac{1}{5^n} \right) \right]$ $= \frac{9}{2 \times 1} \left( \frac{1}{5^2} \right) + \frac{13}{3 \times 2} \left( \frac{1}{5^3} \right) + \dots + \frac{4N-7}{(N-2)(N-3)} \left( \frac{1}{5^{N-2}} \right)$ $= \frac{4 \cdot 1 + 5}{1 \times 2} \left( \frac{1}{5^2} \right) + \frac{4 \cdot 2 + 5}{2 \times 3} \left( \frac{1}{5^3} \right) + \dots + \frac{4(N-3) + 5}{(N-3)(N-2)} \left( \frac{1}{5^{N-2}} \right)$ $= \sum_{n=1}^{N-3} \left[ \frac{4n+5}{n(n+1)} \left( \frac{1}{5^{n+1}} \right) \right]$ $= \frac{1}{5} - \frac{1}{(N-2)5^{N-2}}$

Qn No.	Solution
12 (i)	By substituting (4,1) into the equations of $C_1$ and $C_2$ , $C_1$ : LHS = $(4)^2 + 16(1-1)^2 = 16 = \text{RHS}$ $C_2$ : LHS = $(4)^2 - 16(1-1)^2 = 16 = \text{RHS}$ (justified)
12 (ii)	
12 (iii)	Volume of revolution of $R$ about $x$ -axis $= \pi \int_0^4 \left[ 1 - \sqrt{\frac{16-x^2}{16}} \right]^2 dx + \pi \int_4^{4\sqrt{2}} \left[ 1 - \sqrt{\frac{x^2-16}{16}} \right]^2 dx$ $= 2.17 \text{ units}^3 \text{ (to 3 s.f.)}$
12 (iv)	$\int_0^2 \sqrt{1-(x-1)^2} dx$ $= \int_{\pi}^0 (\sqrt{1-\cos^2 \theta})(-\sin \theta) d\theta$ $= \int_0^{\pi} (\sin^2 \theta) d\theta$ $= \frac{1}{2} \int_0^{\pi} (1-\cos 2\theta) d\theta$ $= \frac{1}{2} \left[ \theta - \frac{\sin 2\theta}{2} \right]_0^{\pi}$ $= \frac{\pi}{2}$
12 (v)	Area of $S = 2 \int_0^2 x dy$ $= 2 \int_0^2 \sqrt{16-16(y-1)^2} dy$ $= 8 \int_0^2 \sqrt{1-(y-1)^2} dy$ $= 8 \left( \frac{\pi}{2} \right)$ $= 4\pi \text{ units}^2$