VICTORIA JUNIOR COLLEGE
Preliminary Examination
Higher 2

MATHEMATICS **9740/01**
Paper 1
Friday **8am – 11am**

16 September 2016 **3 hours**

READ THESE INSTRUCTIONS FIRST

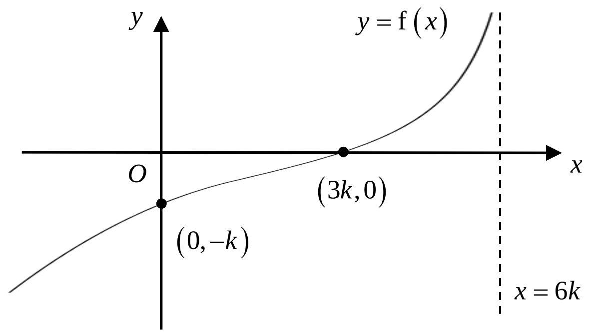
Write your name and CT group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of 6 printed pages

1



The diagram shows the curve with equation $y = \mathrm{f}(x)$, $x < 6k$, $k > \frac{1}{3}$. The curve crosses the x-axis and y-axis at the points $(3k, 0)$ and $(0, -k)$ respectively. Sketch $y = \mathrm{f}(|x| + 1)$. [3]

2 Indicate on a single Argand diagram, the set of points whose complex numbers satisfy the following inequalities

$$\left|\frac{z-6-5\mathrm{i}}{2}\right| \leq 4 \quad \text{and} \quad |2\mathrm{i}-4-z| \geq |z+4-10\mathrm{i}|.$$

Hence, find the least value and greatest value of $\arg(z-6+4\mathrm{i})$. [7]

3 **(a)** Without using a calculator, solve the inequality

$$\frac{4-7x}{x-3} \geq x. \qquad [4]$$

(b) In 2016, Edwin, his father and his grandfather have an average age of 53. In the same year, the sum of one-half of his grandfather's age, one-third of his father's age and one-fourth of Edwin's age is 65. Twenty-two years ago, his grandfather's age was twice the sum of his father's age and his age. What are their respective ages in 2016? [You can assume that Edwin's age in 2016 is more than 22.] [3]

4 The function f is defined by $\mathrm{f}: x \mapsto (x-2)^2 + k,\ x \in \mathbb{R},\ x \leq 2$. It is given that f^{-1} exists.

(i) When $k = 1$,

(a) define f^{-1} in a similar form, [3]

(b) sketch, on a single diagram, the graphs of $y = \mathrm{f}(x)$, $y = \mathrm{f}^{-1}(x)$ and $y = \mathrm{ff}^{-1}(x)$. [3]

(ii) State the set of values of k, such that the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ has no real solutions. [1]

5 A sequence $u_1,\ u_2,\ u_3, \ldots$ is such that $u_1 = 0$ and

$$u_{n+1} = u_n + \frac{2-n^2}{(n+2)!}, \text{ for all } n \geq 1.$$

(i) Show that $u_2 = \frac{1}{6}$, and find the values of u_3 and u_4. [2]

(ii) Hence, give a conjecture for u_n in the form $\frac{n-1}{[\mathrm{f}(n)]!}$, where $\mathrm{f}(n)$ is a function of n to be determined. [1]

(iii) Use the method of mathematical induction to prove your conjecture in part **(ii)** for all positive integers n. [4]

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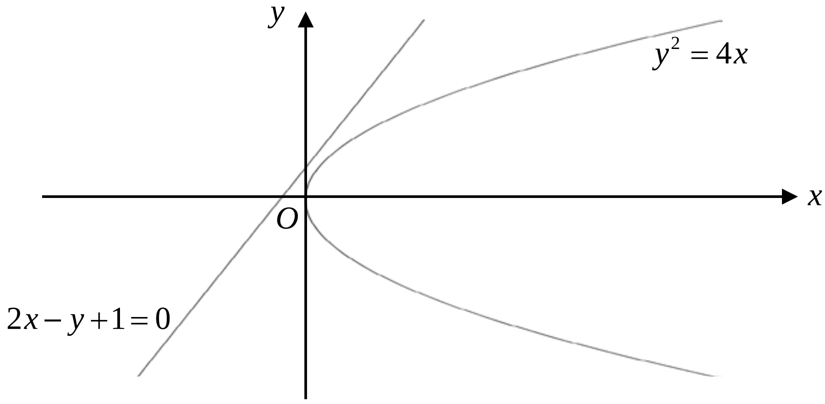

A curve C has equation $y^2 = 4x$ and a line l has equation $2x - y + 1 = 0$. The diagram above shows the graphs of C and l.

$B(b, 2\sqrt{b})$ is a fixed point on C and A is an arbitrary point on l. State the geometrical relationship between the line segment AB and l if the distance from B to A is the least. [1]

Taking the coordinates of A as $(a, 2a+1)$, find an equation relating a and b for which AB is the least. [2]

Deduce that when AB is the least, $(AB)^2 = m\left(2b - 2\sqrt{b} + 1\right)^2$ where m is a constant to be found. Hence or otherwise, find the coordinates of the point on C that is nearest to l, as b varies. [5]

7 **(a)** Differentiate xe^{x^3} with respect to x. Hence, find $\int x^2\left(1+3x^3\right)e^{x^3}\,dx$. [4]

(b) The variables x and y are related by the differential equation

$$\frac{dy}{dx}=\frac{\sec^2 x}{2\sec^2 x+4\tan x+7}.$$

Using the substitution $u=\tan x$, find the general solution of the differential equation. [5]

8 **(i)** Use the method of differences to find, in terms of n,

$$\sum_{r=2}^{n}\ln\left[\frac{r(r+2)}{(r+1)^2}\right].$$ [4]

(ii) Give a reason why the series is convergent and state the sum to infinity. [2]

(iii) Given $\sum_{r=2}^{13}\ln\left[\frac{(2r)(2r+4)}{(r+1)^2}\right]=\ln\left(\frac{p}{q}\right)$, where p and q are integers and $\frac{p}{q}$ is in the simplest form, find the values of p and q. [3]

9 **(i)** Sketch the graph with equation $x^2+(y-r)^2=r^2$, where $r>0$ and $y\le r$. [2]

A hemispherical bowl of fixed radius r cm is filled with water. Water drains out from a hole at the bottom of the bowl at a constant rate.

Use your graph in part **(i)** to show that when the depth of water is h cm (where $h\le r$), the volume of water in the bowl is given by

$$V=\frac{\pi h^2}{3}(3r-h),$$ [3]

(ii) Given that a full bowl of water would become empty in 24 s, find the rate of decrease, in terms of r and h, of the depth of water in the bowl at the instant when the depth of water is h cm. [3]

(iii) Without any differentiation, determine, in terms of r, the slowest rate at which the depth of water is decreasing. [1]

10 The equations of planes p_1 and p_2 are

$$x - 5y + 2z = 13,$$
$$-2x + y + 5z = 1,$$

respectively.

(i) Find the acute angle between p_1 and p_2. [2]

The planes p_1 and p_2 intersect in a line l.

(ii) Find a vector equation of l. [2]

The plane p_3 is perpendicular to both p_1 and p_2. The three planes p_1, p_2 and p_3 intersect at the point $(a, 0, b)$, where a and b are constants.

(iii) Show that $a = 7$ and $b = 3$. [2]

The plane Π is parallel to p_3 and the distance between Π and p_3 is $4\sqrt{11}$ units.

(iv) Find the two possible cartesian equations of Π. [4]

11 (a) An arithmetic progression which consists of $2n$ terms has first term a and common difference d. The third, fifth and twelfth terms of the arithmetic progression are also three distinct consecutive terms of a geometric progression. Find the sum of the even-numbered terms, i.e. the 2^{nd}, 4^{th}, … , $(2n)^{th}$ terms, of the arithmetic progression in terms of a and n. [5]

(b) To renovate his new HDB flat, Douglas is considering taking up a bank loan of $\$40,000$ from Citybank on 1st July 2016. The bank charges a monthly interest of 0.5% on the outstanding amount owed at the end of each month.

Douglas will pay a fixed amount, $\$x$, to the bank at the beginning of each month, starting from September 2016.

(i) Taking July 2016 as the 1st month, show that the amount of money owed at the beginning of the 5th month is

$$1.005^4(40000) - 200x\left(1.005^3 - 1\right).$$ [3]

(ii) If Douglas wishes to pay up his loan within 5 years, find the minimum amount of each monthly repayment. [2]

(iii) Using the value found in part **(ii)**, calculate the interest (to the nearest dollar) that Citybank has earned in total from Douglas's loan at the end of his last repayment. [2]

12 The curve C has equation $y=\frac{\mathrm{f}(x)}{x+a}$, where $\mathrm{f}(x)$ is a quadratic expression, a is a constant and $a \neq \pm 3$. It is given that the coordinates of the points of intersection of C with the x-axis are $(3,0)$ and $(-3,0)$, and the equation of the oblique asymptote is $y=\frac{1}{2}x+1$.

(i) Find $\mathrm{f}(x)$, and show that $a=-2$. [5]

(ii) Sketch C, indicating clearly the equations of the asymptotes, and the coordinates of the points of intersection of C with the x- and y-axes. [2]

A tangent to C is parallel to the line $y=x+2$. Find the possible equations of this tangent, leaving your answer in an exact form. [5]

[End of Paper]