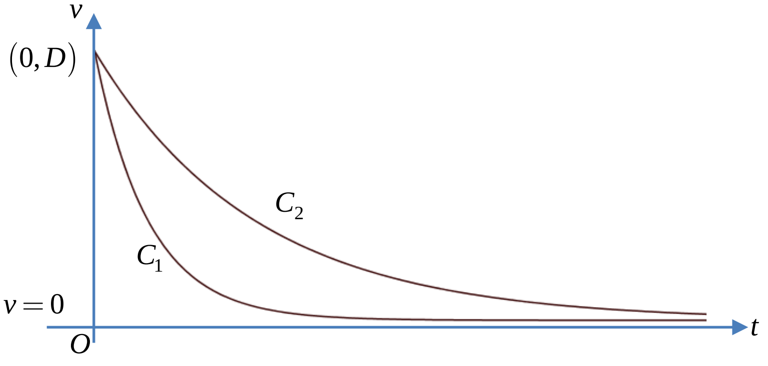
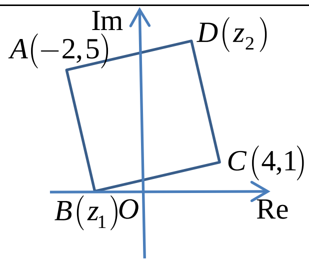
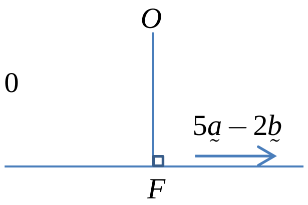
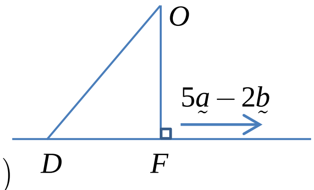
2016 VJC JC2 Prelim Paper 2 Solutions/Comments

Qn	Solution
1i	Since speed is decreasing and v is positive, $\frac{\mathrm{d}v}{\mathrm{d}t}=-kv,\quad$ where k is a positive constant $\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}t}=-k$ $\int\frac{1}{v}\,\mathrm{d}v=\int -k\,\mathrm{d}t$ $\ln v=-kt+C\quad \because v>0$ $v=B\mathrm{e}^{-kt}$ When t = 0s, $v = D$ m s^{-1} $B = D$ Let $k = p$, hence $v=D\mathrm{e}^{-pt}$, where p is a positive constant.
ii	 Stretch C_1 parallel to the t-axis, factor e^2, v-axis is invariant.
2	$y=\sqrt{\mathrm{e}^x\cos^2 x}$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{e}^x\cos^2 x-2\mathrm{e}^x\sin x\cos x}{2\sqrt{\mathrm{e}^x\cos^2 x}}$ $=\frac{y^2-\mathrm{e}^x\sin 2x}{2y}$ $2y\frac{\mathrm{d}y}{\mathrm{d}x}=y^2-\mathrm{e}^x\sin 2x$ **Alternative Solution** $y=\sqrt{\mathrm{e}^x\cos^2 x}$ $y^2=\mathrm{e}^x\cos^2 x$ $2y\frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^x\cos^2 x-2\mathrm{e}^x\sin x\cos x$ $=y^2-\mathrm{e}^x\sin 2x$
i	$2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2y\frac{\mathrm{d}y}{\mathrm{d}x}-\mathrm{e}^x\sin 2x-2\mathrm{e}^x\cos 2x$ When x = 0, $y=\sqrt{\mathrm{e}^0\cos^2 0}=1$

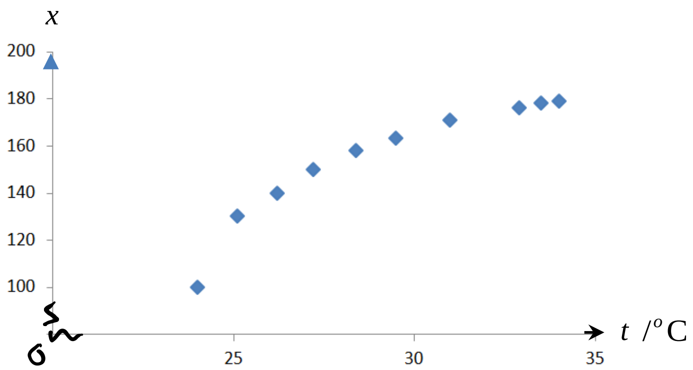
Qn	Solution
	$2\frac{\mathrm{d}y}{\mathrm{d}x}=1-0\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}$ $2\left(\frac{1}{2}\right)^2+2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2\left(\frac{1}{2}\right)-0-2\Rightarrow\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\frac{3}{4}$ $y=1+\frac{1}{2}x-\frac{3}{4}\left(\frac{x^2}{2!}\right)+\ldots$ $=1+\frac{1}{2}x-\frac{3}{8}x^2+\ldots$
ii	$\frac{1}{\sqrt{\mathrm{e}^x\cos^2 x}}=\left(1+\frac{1}{2}x-\frac{3}{8}x^2+\ldots\right)^{-1}$ $=1+(-1)\left(\frac{1}{2}x-\frac{3}{8}x^2+\ldots\right)+\frac{(-1)(-2)}{2!}\left(\frac{1}{2}x+\ldots\right)^2+\ldots$ $=1-\frac{1}{2}x+\frac{3}{8}x^2+\frac{1}{4}x^2+\ldots$ $=1-\frac{1}{2}x+\frac{5}{8}x^2+\ldots$
3a	From the diagram, $\arg(4+\mathrm{i}-z_1)+\frac{\pi}{2}=\arg(-2+5\mathrm{i}-z_1)$ $\mathrm{i}(4+\mathrm{i}-z_1)=(-2+5\mathrm{i}-z_1)$ $4\mathrm{i}-1-\mathrm{i}z_1=-2+5\mathrm{i}-z_1$ $(1-\mathrm{i})z_1=-1+\mathrm{i}$ $z_1=-1$ 
	Midpoint of AC is $\left(\frac{-2+4}{2},\frac{5+1}{2}\right)=(1,3)$ Let $z_2=x+\mathrm{i}y$ Since the diagonals of a square bisect other, Midpoint of BD is $(1,3)$ $\left(\frac{x-1}{2},\frac{y+0}{2}\right)=(1,3)$ $\therefore x=3,\ y=6$ $z_2=3+6\mathrm{i}$
bi	Let $z=\mathrm{e}^{\mathrm{i}\theta}\Rightarrow z^*=\mathrm{e}^{-\mathrm{i}\theta}=\frac{1}{z}$ $z^5=z^{-1}$ $z^6=1$ **Alternatively** $z^5=z^*$ $z^6=zz^*=\|z\|^2$ $z^6=1$
	$z^6=\mathrm{e}^{2k\pi\mathrm{i}},k\in\mathbb{Z}$ $z=\mathrm{e}^{\frac{k\pi\mathrm{i}}{3}},k=0,1,2,3,4,5$ $=1,\mathrm{e}^{\frac{\pi}{3}\mathrm{i}},\mathrm{e}^{\frac{2\pi}{3}\mathrm{i}},-1,\mathrm{e}^{\frac{4\pi}{3}\mathrm{i}},\mathrm{e}^{\frac{5\pi}{3}\mathrm{i}}$

Qn	Solution
ii	Since $0 < \arg(z) < \frac{\pi}{2}$, $z = \mathrm{e}^{\frac{\pi}{3}\mathrm{i}} \Rightarrow z^k = \mathrm{e}^{\frac{k\pi}{3}\mathrm{i}}$ $\frac{(1+\mathrm{i})}{z^k} = \sqrt{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}.\mathrm{e}^{-\frac{k\pi}{3}\mathrm{i}} = \sqrt{2}\mathrm{e}^{\left(\frac{\pi}{4}-\frac{k\pi}{3}\right)\mathrm{i}}$ If $\frac{(1+\mathrm{i})}{z^k}$ is purely imaginary, $\frac{\pi}{4}-\frac{k\pi}{3} = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$ Since k is positive, $\frac{\pi}{4}-\frac{k\pi}{3} = -\frac{\pi}{2}, -\frac{3\pi}{2}, \ldots$ $\frac{k\pi}{3} = \frac{3\pi}{4}, \frac{7\pi}{4}, \ldots$ Smallest positive k when $\frac{k\pi}{3} = \frac{3\pi}{4}$ Smallest positive $k = \frac{9}{4}$
4a	$\overrightarrow{OC} = \frac{1}{3}(2\underset{\sim}{a}+\underset{\sim}{b}), \overrightarrow{OD} = \frac{3}{5}\underset{\sim}{b}$ $\overrightarrow{CD} = \frac{3}{5}\underset{\sim}{b} - \frac{1}{3}(2\underset{\sim}{a}+\underset{\sim}{b}) = -\frac{2}{15}(5\underset{\sim}{a}-2\underset{\sim}{b})$ Since line m passes through D and is parallel to CD, $\underset{\sim}{r} = \overrightarrow{OD} + \mu\overrightarrow{CD}$ $= \frac{3}{5}\underset{\sim}{b} + \frac{2}{15}\mu(2\underset{\sim}{b}-5\underset{\sim}{a})$ $= \frac{3}{5}\underset{\sim}{b} - \frac{2}{15}\mu(5\underset{\sim}{a}-2\underset{\sim}{b})$ $\underset{\sim}{r} = \frac{3}{5}\underset{\sim}{b} + \lambda(5\underset{\sim}{a}-2\underset{\sim}{b}), \lambda \in \mathbb{R}$ Equation of m is $\underset{\sim}{r} = \frac{3}{5}\underset{\sim}{b} + \lambda(5\underset{\sim}{a}-2\underset{\sim}{b}), \lambda \in \mathbb{R}$.
	F is a point on m $\therefore \overrightarrow{OF} = \frac{3}{5}\underset{\sim}{b} + \lambda(5\underset{\sim}{a}-2\underset{\sim}{b})$ for a value of λ  $\overrightarrow{OF}$ is perpendicular to $l \Rightarrow \overrightarrow{OF}\cdot(5\underset{\sim}{a}-2\underset{\sim}{b}) = 0$ $\Rightarrow \left[\frac{3}{5}\underset{\sim}{b} + \lambda(5\underset{\sim}{a}-2\underset{\sim}{b})\right]\cdot(5\underset{\sim}{a}-2\underset{\sim}{b}) = 0$ $\Rightarrow 3(\underset{\sim}{a}\cdot\underset{\sim}{b}) - \frac{6}{5}(\underset{\sim}{b}\cdot\underset{\sim}{b}) + \lambda\left[25(\underset{\sim}{a}\cdot\underset{\sim}{a}) - 20(\underset{\sim}{a}\cdot\underset{\sim}{b}) + 4(\underset{\sim}{b}\cdot\underset{\sim}{b})\right] = 0$ $\Rightarrow 3(\underset{\sim}{a}\cdot\underset{\sim}{b}) - \frac{6}{5}\lvert\underset{\sim}{b}\rvert^2 + \lambda\left[25\lvert\underset{\sim}{a}\rvert^2 - 20(\underset{\sim}{a}\cdot\underset{\sim}{b}) + 4\lvert\underset{\sim}{b}\rvert^2\right] = 0$ Since $\underset{\sim}{a}\cdot\underset{\sim}{b} = \lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{b}\rvert\cos 60^\circ = 2\times5\times\frac{1}{2} = 5$ $\therefore 3(5) - \frac{6}{5}(5)^2 + \lambda\left[25(2)^2 - 20(5) + 4(5)^2\right] = 0$ $\Rightarrow \lambda = \frac{3}{20}$

Qn	Solution
	$\therefore \overrightarrow{OF} = \frac{3}{5}\underset{\sim}{b} + \frac{3}{20}\left(5\underset{\sim}{a} - 2\underset{\sim}{b}\right) = \frac{3}{20}\left(5\underset{\sim}{a} + 2\underset{\sim}{b}\right)$
	Alternative Method  $\overrightarrow{DF} = \left(\overrightarrow{DO} \cdot \frac{5\underset{\sim}{a} - 2\underset{\sim}{b}}{\left\|5\underset{\sim}{a} - 2\underset{\sim}{b}\right\|}\right)\frac{5\underset{\sim}{a} - 2\underset{\sim}{b}}{\left\|5\underset{\sim}{a} - 2\underset{\sim}{b}\right\|}$ $= \frac{1}{\left\|5\underset{\sim}{a} - 2\underset{\sim}{b}\right\|^2}\left(-\frac{3}{5}\underset{\sim}{b} \cdot \left(5\underset{\sim}{a} - 2\underset{\sim}{b}\right)\right)\left(5\underset{\sim}{a} - 2\underset{\sim}{b}\right)$ $= \frac{-3\underset{\sim}{a} \cdot \underset{\sim}{b} + \frac{6}{5}\left\|\underset{\sim}{b}\right\|^2}{\left(5\underset{\sim}{a} - 2\underset{\sim}{b}\right) \cdot \left(5\underset{\sim}{a} - 2\underset{\sim}{b}\right)}\left(5\underset{\sim}{a} - 2\underset{\sim}{b}\right)$ $= \frac{-3(5) + \frac{6}{5}(5)^2}{25\left\|\underset{\sim}{a}\right\|^2 - 20\left(\underset{\sim}{a} \cdot \underset{\sim}{b}\right) + 4\left\|\underset{\sim}{b}\right\|^2}\left(5\underset{\sim}{a} - 2\underset{\sim}{b}\right)$ $= \frac{15}{25(2)^2 - 20(5) + 4(5)^2}\left(5\underset{\sim}{a} - 2\underset{\sim}{b}\right)$ $= \frac{3}{20}\left(5\underset{\sim}{a} - 2\underset{\sim}{b}\right)$ $\therefore \overrightarrow{OF} = \frac{3}{5}\underset{\sim}{b} + \frac{3}{20}\left(5\underset{\sim}{a} - 2\underset{\sim}{b}\right) = \frac{3}{20}\left(5\underset{\sim}{a} + 2\underset{\sim}{b}\right)$
b	The equation of the plane π is $3x + 2y + 5z = 45$. $(p, p, 0)$ lies in $\pi \Rightarrow 3p + 2p + 0 = 45 \Rightarrow p = 9$ $\begin{pmatrix}2\\q\\0\end{pmatrix}$ is perpendicular to $\begin{pmatrix}3\\2\\5\end{pmatrix} \Rightarrow \begin{pmatrix}2\\q\\0\end{pmatrix} \cdot \begin{pmatrix}3\\2\\5\end{pmatrix} = 0$ $6 + 2q = 0 \Rightarrow q = -3$ Since $\underset{\sim}{v}$ is perpendicular to both $\underset{\sim}{u}$ and $\underset{\sim}{n}$, $\underset{\sim}{u} \times \underset{\sim}{n} = \begin{pmatrix}3\\2\\5\end{pmatrix} \times \begin{pmatrix}2\\-3\\0\end{pmatrix} = \begin{pmatrix}15\\10\\-13\end{pmatrix}$ $\underset{\sim}{r} = \begin{pmatrix}9\\9\\0\end{pmatrix} + \lambda\begin{pmatrix}2\\-3\\0\end{pmatrix} + \mu\begin{pmatrix}15\\10\\-13\end{pmatrix}, \lambda, \mu \in \mathbb{R}$ **Alternative method to find** $\underset{\sim}{v}$ Let $\underset{\sim}{v} = \begin{pmatrix}x\\y\\z\end{pmatrix}$ $\begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}3\\2\\5\end{pmatrix} = 0$ and $\begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}2\\-3\\0\end{pmatrix} = 0$

Qn	Solution
	$3x + 2y + 5z = 0$ and $2x - 3y = 0$ $x = -\frac{15}{13}z,\ y = -\frac{10}{13}z,\ z = z$ Let $z = 13$ (any non-zero number will work) $\mathbf{v} = \begin{pmatrix} -15 \\ -10 \\ 13 \end{pmatrix}$
5i	He will not get to survey the students who do not go to the school gymnasium. Hence, the sample obtained is biased.
ii	He can obtain a numbered list of all the students (labelled 1 to N) in the school. Using a random number generator, he generates 30 **distinct** numbers. He will survey the students corresponding the numbers generated. **Alternatively.** Let the total number of students be N Sampling interval = $\frac{N}{30}$ He can obtain a numbered list of all the students (labelled 1 to N) in the school. Using a random number generator, select a starting number k where $1 \le k \le \frac{N}{30}$. He can interview the students corresponding to the numbers $k,\ k + \frac{N}{30}, k + \frac{N}{15}, \ldots, k + \frac{29N}{30}$.
6i	Number of 4-digit numbers = $5^4 = 625$
ii	Case1: Starts with 1 1 2 No. of ways = 2(3!) = 12 1 __ __ __ 6 Case 2: starts with 2 No. of ways = 3! = 6 2 __ __ __ 6 Total number of ways = 18
iii	Case 1: XXXXYZ No. of ways = ${}^5C_3({}^3C_1) = 30$ Case 2: XXXYYZ No. of ways = ${}^5C_3({}^3C_1)({}^2C_1) = 60$ Case 3: XXYYZZ No. of ways = ${}^5C_3 = 10$ Total number of ways = 100
7i	P(Mr Wong is correct) = $\left(1 - \frac{{}^{15}C_3}{{}^{25}C_3}\right) \times \frac{{}^3C_2}{{}^5C_2} = 0.24065 = 0.241$
ii	P(Mr Tan is correct) = $\frac{{}^{10}C_1 \times {}^8C_1 \times {}^7C_1}{{}^{25}C_3} \times \frac{{}^3C_1 \times {}^2C_1}{{}^5C_2} = \frac{84}{575} = 0.146$ <ins>Alternative method:</ins> P(Mr Tan is correct) $= \frac{10 \times 8 \times 7}{25 \times 24 \times 23} \times 3! \times \frac{3 \times 2}{5 \times 4} \times 2!$

Qn	Solution
	$= \dfrac{84}{575} = 0.146$
iii	P(Mr Wong's guess is right, given that Mr Tan's guess is wrong) $= \dfrac{\text{P(Mr Wong is correct and Mr Tan is wrong)}}{\text{P(Mr Tan is wrong)}}$ $= \dfrac{0.24065}{1-0.14609}$ $= 0.282$
8i	Let T be the total number of refrigerators sold in a 4-week period. $T \sim \text{Po}((1.3 + 1.1) \times 4)$ $T \sim \text{Po}(9.6)$ $\text{P}(T \geq 10) = 1 - \text{P}(T \leq 9) = 0.49114 = 0.491$ (3sf)
ii	Let X be number of good periods out of 52. $X \sim \text{B}(52, 0.491)$ or $X \sim B(52,\ 0.49114)$ Since $np = 25.532 > 5$ and $np(1-p) = 26.468 > 5$ $X \sim \text{N}(25.532, 12.996)$ approx. or $X \sim N(25.539, 12.996)$ approx. $\text{P}(25 < X \leq 32) = \text{P}(25.5 < X \leq 32.5) = 0.477$ (or 0.478)
iii	We need to assume that the sales of **all** the refrigerators are independent of one another. We also need to assume that the **average rate** of refrigerators being sold is constant. The first assumption may not hold as the two brands of refrigerator are in the same price range and they can be competing in terms of sales. OR The average rate of refrigerators sold is unlikely to be a constant due to sale, festive seasons, economic conditions etc.
9	Let A kg and B kg be masses of a randomly chosen grade A and grade B durian respectively. $A \sim \text{N}(1.96, 0.24^2)$ and $B \sim \text{N}(1.00, \sigma^2)$ $\text{P}(B > 0.8) = 0.95$ $\text{P}\left(Z > \dfrac{0.8-1.00}{\sigma}\right) = 0.95$ $\text{P}\left(Z \leq \dfrac{0.8-1.00}{\sigma}\right) = 0.05$ $\dfrac{-0.2}{\sigma} = -1.64485 \Rightarrow \sigma = 0.12159 \approx 0.122$
i	$A \sim \text{N}(1.96, 0.24^2)$ and $B \sim \text{N}(1.00, \sigma^2)$ $\overline{A} \sim N\left(1.96, \dfrac{0.24^2}{50}\right)$ and

Qn	Solution
	$2B \sim N(2.00, 2^2(0.122^2))$ or $2B \sim N(2.00, 2^2(0.12159)^2)$ $\overline{A} - 2B \sim N(-0.04, 0.060688)$ or $\overline{A} - 2B \sim N(-0.04, 0.060290)$ $P(\overline{A} - 2B > 0) = 0.436$ (or 0.435) Central limit theorem is not needed because the masses of grade *A* durians follow a normal distribution.
ii	Let *Y* be the number of grade *B* durians with a mass of more than 0.8 kg out of 50 durians. $Y \sim B(50, 0.95)$ $np = 50 \times 0.95 = 47.5 > 5$ and $n(1-p) = 50 \times 0.05 = 2.5 < 5$ Let *Y'* be the number of grade *B* durians with a mass $\leq$ 0.8 kg out of 50 durians. $Y' \sim Po(2.5)$ approx. $P(Y > 47) = P(50 - Y' > 47)$ $= P(Y' \leq 2)$ $= 0.544$
10i	$\bar{t}$ and $\bar{x}$ $\bar{t} = 29.18$, $\bar{x} = 154.5$ Hence, (29.18, 154.5) lies on the regression line *x* on *t*.
ii	 $r = 0.934$ (3s.f.)
iii	From the scatter diagram, *x* increases by decreasing amounts as *t* increases. Hence, a quadratic model might be more appropriate.
iv	By GC, $a = -0.673$ (3sf), $b = 179$ (3sf)
v	Substituting $t = 31.0$, $x = -0.67342(34.2 - 31.0)^2 + 179.28$ $= 172.388$

<table>
<tr><td>Qn</td><td>Solution</td></tr>
<tr><td></td><td>Expected number of cups of ice cream sold is 172.</td></tr>
<tr><td>11</td><td>$H_0: \mu = 400$
$H_1: \mu \neq 400$
Level of significance: 5%
Test Statistic: When H_0 is true, $T = \frac{\overline{X} - 400}{S / \sqrt{5}}$
Computation: $\nu = 5 - 1 = 4$.
By GC, $\overline{x} = 392.34, s = 12.971$, p-value = 0.257 (3sf)

Conclusion: Since p-value = 0.257 > 0.05, H_0 is not rejected at 5% level of significance. So there is insufficient evidence to conclude that the claim is invalid.

It is assumed that the masses of loaves of “Gardener” wholemeal bread follow a normal distribution.</td></tr>
<tr><td></td><td>$\overline{x} = 400 - \frac{102.4}{50} = 397.952$
$s^2 = \frac{1}{49}\left(8030.2 - \frac{(-102.4)^2}{50}\right) = 159.60$

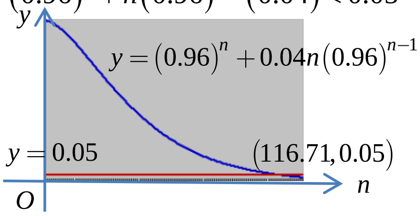
$H_0: \mu = 400$
$H_1: \mu \neq 400$
Level of significance: k%
Test Statistic: When H_0 is true, $Z = \frac{\overline{X} - 400}{\sqrt{159.6017306} / \sqrt{55}}$
Computation:
By GC, $\overline{x} = 397.952$, p-value = 0.252 (3sf)

For H_0 to be rejected at k% level of significance,
p-value $\leq \frac{k}{100} \Rightarrow k \geq 25.2$
Set of values = $\{k \in \mathbb{R} : k \geq 25.2\}$

“k% significance level” in this context means there is a probability of $\frac{k}{100}$ (or k%) that the test will conclude that the mean mass of “Gardener” wholemeal bread is not 400g, when it is actually 400g.</td></tr>
<tr><td></td><td>Let Y be the number of wrong conclusions out of n hypothesis tests
$Y \sim \mathrm{B}(n, 0.04)$
$P(Y \leq 1) < 0.05$

By GC,
<table><tr><td>n</td><td>$P(Y \leq 1)$</td></tr><tr><td>116</td><td>0.05121</td></tr><tr><td>117</td><td>0.04952</td></tr></table>Least n = 117

Alternatively</td></tr>
</table>

Qn	Solution
	$P(Y \le 1) < 0.05$ $\left(0.96\right)^n + n\left(0.96\right)^{n-1}\left(0.04\right) < 0.05$  Least n = 117